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Dynamic Single Facility Location under Cumulative Customer Demand

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Abstract. Dynamic facility location problems aim at placing one or more valuable resources over a planning horizon to meet customer demand. Existing literature commonly assumes that customer demand quantities are defined independently for each time period. In many planning contexts, however, unmet demand carries over to future time periods. Unmet demand at some time periods may therefore affect decisions of subsequent time periods. This work studies a novel location problem, where the decision maker relocates a single temporary facility over time to capture cumulative customer demand. We propose two mixed-integer programming models for this problem, and show that one of them has a tighter continuous relaxation and allows the representation of more general customer demand behaviour. We characterize the computational complexity for this problem, and analyze which problem characteristics result in NP-hardness. We then propose an exact branch-and-Benders-cut method, and show how optimality cuts can be computed efficiently through an analytical procedure. Computational experiments show that our method is approximately 30 times faster than solving the tighter formulation directly. Our results also quantify the benefit of accounting for cumulative customer demand within the optimization framework, since the corresponding planning solutions perform much better than those obtained by ignoring cumulative demand or employing myopic heuristics.

Keywords: Facility location, multi-period planning, cumulative customer demand.

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1 Introduction

Dynamic facility location problems are a classical family of combinatorial problems that aim at placing one or more valuable resources over a planning horizon (Nickel and Saldanha-da Gama, 2019). For example, energy suppliers may want to place charging stations while facing different levels of electric vehicle adoption growth throughout time depending on their location decisions (Lamontagne et al., 2023). Similarly, humanitarian organizations often locate relief facilities and might need to account for demand shifts over time to accommodate future circumstances (Alizadeh et al., 2021). In broad terms, the literature often designates the valuable resource to be located as a *facility*, the entity seeking services at one of these facilities as a *customer*, the quantity of service sought by a customer at a facility as *demand*, and key moments in time where location decisions are assumed to be made as *time periods* (Laporte et al., 2019).

In most dynamic location problems, customer demand is fixed for each time period and must be served while optimizing a specific performance measure (*e.g.*, Ballou, 1968; Wesolowsky and Truscott, 1975; Van Roy and Erlenkotter, 1982). In the case where customer demand cannot be served completely (*e.g.*, due to the lack of sufficient resources or technical restrictions), most works explicitly or implicitly maximize captured demand (*e.g.*, Gunawardane, 1982; Marín et al., 2018; Vatsa and Jayaswal, 2021). Unmet demand is typically assumed to vanish, thus not impacting location decisions of subsequent time periods. In many planning contexts, however, unmet demand carries over to future time periods, until eventually being served. This is the case, for example, in temporary retail (Rosenbaum et al., 2021), where pop-up stores satisfy customer demand for seasonal or luxury goods accumulated since the last activation; in healthcare campaigns (Qi et al., 2017), where mobile units need to treat patients accumulated since the last visit; and in humanitarian logistics (Daneshvar et al., 2023), where emergency responders need to fulfill requests accumulated since the last check-in.

In this paper, we consider the location problem faced by a service provider that moves a temporary facility over time to capture customer demand. Such a temporary facility could be, for example, a temporary retail store (Rosenbaum et al., 2021; Clothiers, 2024), a mobile healthcare unit (Büsing et al., 2021; Dubinski, 2021), or an embassy providing mobile consulate services (Nica and Moraru, 2020; Nzioka, 2024). Customers may decide that the location of the temporary facility is sufficiently close to obtain service (*e.g.*, for retail goods, medical supplies, and official documents). In this case, the entire accumulated demand of that customer is served. In contrast, if the customer is unwilling to attend the location of the temporary facility, its unmet demand remains critical and builds up while waiting for service. The provider therefore needs to be strategic about when and where to install its temporary facility.

Explicitly considering the accumulation of unmet demand may result not only in a different sequence of locations to visit, but also in a higher satisfaction of demand and/or in a higher collection of rewards. We highlight that cumulative customer demand may also appear in other multi-period planning problems such as vehicle routing (*e.g.*, if a customer is visited at a later stage of the route rather than at the beginning, the demand of that customer may have already increased), network design and distribution networks (*e.g.*, customer demand may increase as long as said customer is not connected to the distribution network), and production scheduling (*e.g.*, products scheduled for later time periods may have a higher demand than initially expected). However, to the best of our knowledge, the literature on cumulative customer demand for other multi-period planning problems is quite sparse, and nonexistent for facility location problems.

To fill this gap, we investigate a novel location problem named the Dynamic Single Facility Location Problem under Cumulative Customer Demand (CDSFLP-CCD). We concentrate on the deterministic case, where parameters modelling cumulative demand are either known or can be

sufficiently well-estimated in advance. More specifically, we contribute to the literature on location problems as follows:

1. We introduce a novel multi-period deterministic location problem, referred to as CDSFLP-CCD, where the decision maker relocates a single temporary facility over time to capture cumulative customer demand.
2. We propose two mixed-integer programming formulations for the CDSFLP-CCD: an intuitive nonlinear formulation that can be linearized, and an integer linear reformulation that *(i)* provides a tighter continuous relaxation and *(ii)* allows the representation of more general customer demand behaviour.
3. We characterize the computational complexity for some special cases of the CDSFLP-CCD, and provide insights on which problem characteristics render it *(i)* NP-hard or *(ii)* even inapproximable.
4. We present a 2-approximate algorithm for a special NP-hard case. Applied as a heuristic to the CDSFLP-CCD, this algorithm tends to find high-quality solutions (in our computational experiments, on average, within 2% of the optimal solution), thus being a relevant approach for tackling large-scale instances.
5. We propose *(i)* an exact Benders decomposition for our reformulation implemented in a branch-and-Benders-cut fashion, where optimality cuts are computed through an analytical procedure, and *(ii)* myopic heuristics to obtain solutions of reasonable quality when the decision maker ignores cumulative customer demand.
6. We compare the two proposed formulations in terms of solution times when fed to an off-the-shelf solver, and validate the practical advantage of having a tighter continuous relaxation. We also show the efficiency of the proposed Benders decomposition, which *(i)* is 30 times faster than solving the reformulation directly with a solver and *(ii)* proves five times smaller optimality gaps within the same time limit.
7. We highlight the benefit of accounting for cumulative customer demand within the optimization framework, as myopic heuristics provide solutions that do not perform well whenever customer demand is, in fact, cumulative (in our computational experiments, on average, within 8% of the optimal solution).

The remainder of this paper is organized as follows. Section 2 discusses the literature on dynamic location problems, paying close attention to how they model customer demand. Section 3 defines the CDSFLP-CCD and presents its two-mixed-integer programming formulations. Section 4 summarizes theoretical results concerning the computational complexity of the CDSFLP-CCD, and Section 5 describes the proposed exact and heuristic methods. Section 6 presents computational results, and Section 7 concludes with final remarks.

2 Literature Review

In this section, we discuss relevant works on multi-period location problems, with emphasis on the customer demand modelling. We refer the reader to Nickel and Saldanha-da Gama (2019) for a broader view of the related literature. For the sake of clarity, whenever we mention customer demand, we refer to the *quantity* of commodities or service sought by a customer at one or more facilities.

Ballou (1968) studies the location of a single warehouse throughout a planning horizon, where said facility must serve customer demand completely at each time period (*i.e.*, there is no unmet demand). Note that said facility is able to serve all customers from the chosen location. Some authors expand this model while still restricted to a single facility (*e.g.*, Wesolowsky, 1973), while others allow multiple facilities (*e.g.*, Wesolowsky and Truscott, 1975; Sweeney and Tatham, 1976; Van Roy and Erlenkotter, 1982; Hormozi and Khumawala, 1996). These works still require, however, the decision maker to serve customer demand completely at each time period of the planning horizon, whilst optimizing some performance measure.

Gunawardane (1982) is among the first to relax this requirement due to technical restrictions. For example, clinics (facilities) may not be able to reach some patients (customers) that are out of reach (Vatsa and Jayaswal, 2021), or individuals (customers) may not buy an electric vehicle if charging stations (facilities) are not sufficiently convenient (Lamontagne et al., 2023). In these contexts, the decision maker implicitly or explicitly maximizes captured demand (*e.g.*, Dell’Olmo et al., 2014; Zarandi et al., 2013; Marín et al., 2018; Alizadeh et al., 2021; Vatsa and Jayaswal, 2021; Lamontagne et al., 2023). These works assume, however, that unmet demand at some time period simply vanishes, thus not impacting location decisions of subsequent time periods.

From a modelling perspective, the literature often represents customer demand as a coefficient in the objective function or a constant on the right-hand side of constraints in mixed-integer programs. On the deterministic front, these parameters have a fixed value (*e.g.*, Alizadeh et al., 2021). On the stochastic front, these parameters are random variables within stochastic programs (*e.g.*, Marín et al., 2018). We highlight that the realization of the random variable is exogenous (*i.e.*, independent) with respect to location decisions. Therefore, these works do not allow the representation of cumulative customer demand (*i.e.*, the realization of customer demand as cumulative based on past location decisions).

To the best of our knowledge, Qi et al. (2017) is the only work that considers customer demand dependent on past location decisions within a location problem. They study the so-called service-time-related demand, where customer demand starts at an initial value when the mobile facility arrives, and decreases progressively until the mobile facility leaves (*i.e.*, customer demand is proportional to the total service time). Their demand behaviour is, however, significantly different from ours because they do not allow customer demand to build up over time, and customers can only be captured once over the planning horizon.

Moreover, to the best of our knowledge, cumulative customer demand has not been considered for other multi-period planning problems with a similar structure. One could naturally see the connection between the CDSFLP-CCD and vehicle routing problems, where a single vehicle represents the temporary facility. However, recent surveys show that existing models cannot account for cumulative demand behaviour (Braekers et al., 2016). For other multi-period planning problems, such as network design and scheduling, the literature remains similarly sparse. To the best of our knowledge, Daneshvar et al. (2023) is the only work accounting for cumulative demand behaviour in the context of humanitarian supply chains. Here, unmet demand for first-aid response resources carries over to future time periods and may spread further if left unmet.

Although the CDSFLP-CCD is deterministic, customer demand ultimately depends on the location decisions and may be interpreted as a special case of endogenous uncertainty, as in certain two-stage stochastic (*e.g.*, Hellemo et al., 2018) and multi-stage stochastic (*e.g.*, Yu and Shen, 2022) programs. However, current approaches consider planning problems that are unnecessarily general and, as such, cannot exploit the structure of our specific planning problem.

3 Mathematical Models

We define the CDSFLP-CCD along with key problem characteristics in Section 3.1 and model it as a mixed-integer linear program in Section 3.2. We then propose a reformulation as an integer linear program in Section 3.3, and highlight formulation properties and extensions in Section 3.4.

3.1 Problem Definition

Consider a service provider that intends to relocate a single temporary facility throughout a planning horizon in order to capture cumulative demand from a set of targeted customers. The provider has available a set of candidate locations, and each location has a reward per unit of captured demand. Each customer is willing to attend only a subset of candidate locations, according to their individual preferences. In addition, each customer has a function that indicates the amount of additional demand at each time period (*e.g.*, obtained through reliable forecast methods), which we refer to as spawning demand. For each customer, the spawning demand accumulates over time. Once a customer is captured by a facility, its entire accumulated demand is assumed to be satisfied. The provider aims to determine the location of the temporary facility at each time period of the planning horizon so as to maximize the total reward obtained from targeted customers.

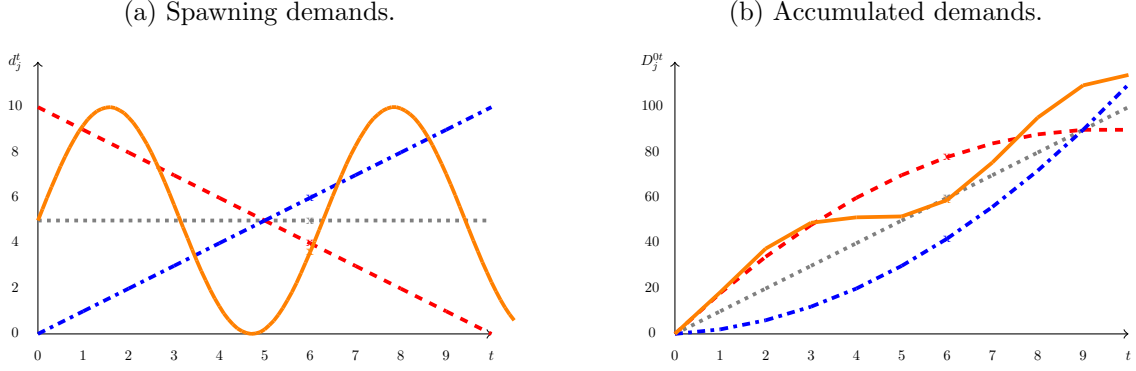
Throughout the rest of this paper, let $\mathcal{I} = \{1, \dots, I\}$ be the set of candidate locations, $\mathcal{J} = \{1, \dots, J\}$ be the set of targeted customers, and $\mathcal{T} = \{1, \dots, T\}$ be the set of time periods. Let also $\mathcal{T}^S = \mathcal{T} \cup \{0\}$ and $\mathcal{T}^F = \mathcal{T} \cup \{T + 1\}$ be the set of time periods with the start period 0 and the final period $T + 1$, respectively. Let $r_i \in \mathbb{R}^+$ be the reward per unit of demand captured at location $i \in \mathcal{I}$. In addition, let $d_j^t \in \mathbb{R}^+$ be the spawning demand of customer $j \in \mathcal{J}$ at time period $t \in \mathcal{T}$, and, for each location $i \in \mathcal{I}$, $a_{ij} \in \{0, 1\}$ be the preference rule of customer j (*i.e.*, 1 if customer j is willing to attend the temporary facility in location i , 0 otherwise). We employ bold letters and sets to denote vectors (*e.g.*, \mathbf{r} and $\{r_i\}_{i \in \mathcal{I}}$ for location rewards).

We refer to the accumulated demand of customer $j \in \mathcal{J}$ at time period $t \in \mathcal{T}$ after being lastly captured at time period $\ell \in \mathcal{T}^S$, $\ell < t$ as $D_j^{\ell t} = \sum_{s=\ell+1}^t d_j^s$. Figure 1 exemplifies different functions of spawning demand over time for a given customer j , as well as their corresponding accumulated counterparts when left unmet since the beginning of the planning horizon (*i.e.*, when customer j is lastly captured at time period $\ell = 0$). This figure exhibits certain trade-offs considered within the CDSFLP-CCD. For example, consider a scenario where the provider can only capture one customer among four that have not been captured since the start period $\ell = 0$ at time period $t = 6$. We refer to them as customers A , B , C , and D with constant, increasing, decreasing and seasonal demands, respectively. If the provider only takes the spawning demand into account, customer B would be preferred because it has the highest spawning demand at time period $t = 6$, as shown in Figure 1a. However, customer B is the worst choice in terms of accumulated demand at time period $t = 6$, as shown in Figure 1b, and customer C should be preferred over customers A , B , and D .

The CDSFLP-CCD has many problem characteristics (*e.g.*, reward structure and preference rules) to represent a wide range of real-world applications. However, the relevance of each problem characteristic depends on the particular application at hand. For example, in some applications, candidate locations may have the same reward per unit of capture demand (*i.e.*, $r_i = R, \forall i \in \mathcal{I}, R \in \mathbb{R}^+$). In this sense, we define the following descriptors for CDSFLP-CCD instances, outlining special cases of the general problem.

Definition 1 (Loyal customers) *A CDSFLP-CCD instance is said to have loyal customers if every customer is willing to attend only one location (*i.e.*, $\sum_{i \in \mathcal{I}} a_{ij} = 1, \forall j \in \mathcal{J}$).*

Figure 1: Examples of spawning demand functions over time for customer j , and their accumulated counterparts, when customer j is lastly captured at time period $\ell = 0$. Constant in dotted gray, increasing in dash-dotted blue, decreasing in dashed red, seasonal in solid orange.



Definition 2 (Flexible customers) A CDSFLP-CCD instance is said to have flexible customers if at least one customer is willing to attend more than one location (i.e., $\exists j \in \mathcal{J}$ such that $\sum_{i \in \mathcal{I}} a_{ij} > 1$).

Definition 3 (Identical rewards) A CDSFLP-CCD instance is said to have identical rewards if every location has the same reward (i.e., $r_i = R, \forall i \in \mathcal{I}, R \in \mathbb{R}^+$).

Definition 4 (Different rewards) A CDSFLP-CCD instance is said to have different rewards if at least two locations have different rewards (i.e., $\exists i, k \in \mathcal{I}, i \neq k$ such that $r_i \neq r_k$).

3.2 Single Index Formulation

Let decision variables $y_i^t \in \{0, 1\}$ be 1 if the provider places the facility at location i at time period t , 0 otherwise. We first propose a formulation that tracks accumulated demand by means of continuous decision variables with a single time index. As such, we refer to this formulation as the Single Index (SI) formulation. Let $c_j^t \in \mathbb{R}^+$ be the accumulated demand of customer j at the beginning of time period t , $b_j^t \in \mathbb{R}^+$ be the accumulated demand of customer j at the end of time period t , and $w_j^t \in \mathbb{R}^+$ be the captured demand of customer j at time period t . The SI Formulation writes as follows:

$$\max_{\mathbf{b}, \mathbf{c}, \mathbf{w}, \mathbf{y}} \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} r_i a_{ij} w_j^t y_i^t \quad (1a)$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} y_i^t \leq 1 \quad \forall t \in \mathcal{T} \quad (1b)$$

$$b_j^0 = 0 \quad \forall j \in \mathcal{J} \quad (1c)$$

$$c_j^t = b_j^{t-1} + d_j^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (1d)$$

$$b_j^t = c_j^t - w_j^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (1e)$$

$$w_j^t = \left(\sum_{i \in \mathcal{I}} a_{ij} y_i^t \right) c_j^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (1f)$$

$$b_j^t \in \mathbb{R}^+ \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T}^S \quad (1g)$$

$$c_j^t \in \mathbb{R}^+ \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (1h)$$

$$w_j^t \in \mathbb{R}^+ \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (1i)$$

$$y_i^t \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}. \quad (1j)$$

Objective Function (1a) maximizes the total reward obtained by capturing customer demand. Constraints (1b) guarantee that the provider installs at most one facility per time period. Constraints (1c)–(1f) ensure proper cumulative demand behaviour over time. In particular, Constraints (1f) determine the captured demand for customer j at time period t depending on whether customer j is captured by the temporary facility and the quantity of accumulated demand. Finally, Constraints (1g)–(1j) define feasible variable domains.

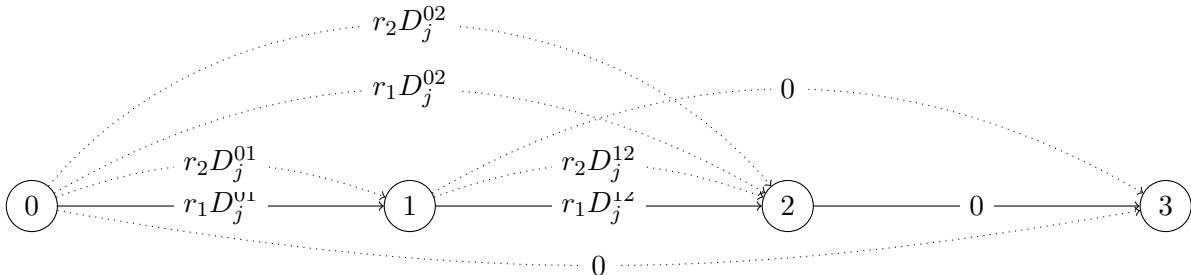
For the sake of simplicity, Formulation (1) is the nonlinear version of the SI Formulation, but we can apply standard techniques to linearize it (see Appendix A). Even though the linearization requires the use of big- M constraints, preliminary results show that off-the-shelf solvers perform better on the linearized version rather than on the nonlinear one. Therefore, we employ the linearized version of the SI Formulation.

3.3 Double Index Formulation

The introduction of big- M constraints to linearize bilinear terms $w_j^t y_i^t$ in Objective Function (1a) and $y_i^t c_i^t$ in Constraints (1f) of the SI Formulation is prone to provide loose continuous relaxation bounds. We therefore propose a reformulation that avoids these terms based on how customers accumulate demand over time.

We specify, for each customer j , a graph $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{A}_j)$, where $\mathcal{N}_j = \mathcal{T} \cup \{0, T + 1\}$, and $\mathcal{A}_j = \{(\ell, t, i) \in \mathcal{T}^S \times \mathcal{T}^F \times \mathcal{I} \mid \ell < t\}$. Each node in \mathcal{N}_j represents a time period t where customer j may be captured by the provider. We assume that customer j must be captured at the start period 0 and at the final period $T + 1$, as these time periods do not contribute to the total reward. Each arc in \mathcal{A}_j represents a connection between time periods ℓ and t through location i , which acts as a label among parallel arcs. If there is flow in arc (ℓ, t, i) , we know that customer j (i) is captured by some location k in time period ℓ , (ii) is not captured between time periods ℓ and t , and (iii) is captured by location i at time period t . By fixing a location sequence \mathbf{y} , the provider implicitly induces a single path between nodes 0 and $T + 1$ in graph \mathcal{G}_j of each customer j , where the sum of the weights $r_i D_j^{\ell t}$ for each arc (ℓ, t, i) in the path gives the total reward obtained from customer j over the planning horizon. Figure 2 exemplifies graph \mathcal{G}_j for some customer j .

Figure 2: Graph \mathcal{G}_j of customer j in an instance with $\mathcal{I} = \{1, 2\}$ and $\mathcal{T} = \{1, 2\}$, and $a_{ij} = 1 \forall i \in \mathcal{I}$, where solid arcs denote the path when the provider installs location 1 at time periods 1 and 2.



We refer to this formulation as Double Index (DI) because decision variables related to the

cumulative demand have two time indexes. Let decision variables $x_{ij}^{\ell t} \in \{0, 1\}$ be 1 if location i captures customer j at time period t after being lastly captured at time period ℓ , 0 otherwise (*i.e.*, 1 if arc (ℓ, t, i) is part of the single path between nodes 0 and $T + 1$ for customer j , 0 otherwise). The DI Formulation writes as follows:

$$\max_{\mathbf{x}, \mathbf{y}} \sum_{t \in \mathcal{T}} \sum_{\substack{\ell \in \mathcal{T}^S: \\ \ell < t}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} r_i D_j^{\ell t} x_{ij}^{\ell t} \quad (2a)$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} y_i^t \leq 1 \quad \forall t \in \mathcal{T} \quad (2b)$$

$$\sum_{\substack{\ell \in \mathcal{T}^S: \\ \ell < t}} x_{ij}^{\ell t} = a_{ij} y_i^t \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (2c)$$

$$\sum_{\substack{s \in \mathcal{T}^S: \\ s < t}} \sum_{i \in \mathcal{I}} x_{ij}^{st} = \sum_{\substack{s \in \mathcal{T}^F: \\ s > t}} \sum_{i \in \mathcal{I}} x_{ij}^{ts} \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (2d)$$

$$\sum_{s \in \mathcal{T}^F} \sum_{i \in \mathcal{I}} x_{ij}^{0s} = 1 \quad \forall j \in \mathcal{J} \quad (2e)$$

$$x_{ij}^{\ell t} \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall \ell \in \mathcal{T}^S, \forall t \in \mathcal{T}^F : \ell < t \quad (2f)$$

$$y_i^t \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}. \quad (2g)$$

Objective Function (2a) maximizes the total reward obtained by capturing customer demand. Constraints (2b) guarantee that the provider installs at most one single facility per time period. Constraints (2c) force the single path of customer j to visit time period t if location i captures said customer. Constraints (2d)–(2e) ensure the construction of a path for customer j . More specifically, Constraints (2e) require this path to begin at the start period 0, and Constraints (2d) require this path to continue through a future time period until the final period $T + 1$. Finally, Constraints (2f)–(2g) define feasible variable domains.

3.4 Properties and Extensions

Once a location sequence \mathbf{y} is chosen by the provider, the values of the remaining components within the SI and DI Formulations are unambiguously fixed, as formalized in Proposition 1:

Proposition 1 *Feasible solutions of the SI and DI Formulations can be solely represented by a location sequence \mathbf{y} , since the remaining components have fixed values computable in polynomial time.*

In what follows, we refer to feasible solutions of the CDSFLP-CCD solely by a location sequence \mathbf{y} or, equivalently, i_1, \dots, i_T , where $i_t = \emptyset$ denotes that there is no facility at time period t . Let $\pi(\mathbf{y}) = \pi(i_1, \dots, i_T)$ be the total reward of location sequence \mathbf{y} . Proposition 1 allows us to relax integrality constraints on variables $x_{ij}^{\ell t}$ in the DI Formulation, as long as variables y_i^t remain binary, and to compute the total reward $\pi(\mathbf{y})$ of location sequence \mathbf{y} in polynomial time.

Although the SI and DI Formulations are equivalent (*i.e.*, they have the same space of feasible and optimal integer solutions), the DI Formulation may be preferred from a theoretical point of view, because it provides a tighter continuous relaxation.

Theorem 1 *The DI Formulation provides a tighter continuous relaxation than the SI Formulation.*

Proof. See Appendix B.1.

We highlight that the DI Formulation is not only tighter, but also allows the representation of more general customer demand behaviour. In fact, it can represent any customer demand behaviour that depends solely on time periods ℓ and t for each customer j . In this sense, we might represent the cumulative demand behaviour studied by Daneshvar et al. (2023), where spawning demand in one time period is defined as a percentage of the unmet demand of the previous time period. Although the SI Formulation can be adapted to account for such type of demand spread, preliminary results show that the resulting SI Formulation yields worse continuous relaxation bounds than the DI Formulation, as this type of demand spread adds complexity to the constraints. Although some theoretical results do not hold for the problem variant with more general customer demand behaviour (namely, the approximation guarantees further discussed in Section 4), we highlight that our solution methods, further presented in Section 5, could be seamlessly employed.

Note that we can naturally expand the definition of the CDSFLP-CCD to include other problem characteristics relevant in particular applications. For example, the reward per unit of demand captured at location i and the preference rule of customer j may vary with time period t (e.g., during winter, the reward may be lower due to heating costs and customers may not be willing to travel as far to obtain service). In this sense, we could employ time-dependent rewards $r_i^t \in \mathbb{R}^+$ and time-dependent preference rules $a_{ij}^t \in \{0, 1\}$ in the SI and DI Formulations, and easily adapt solution methods accordingly.

We might also explicitly account for (time-dependent) maintenance and relocation costs. Let $f_i^t \in \mathbb{R}^+$ be the maintenance cost from installing location i at time period t , and $g_{ki}^t \in \mathbb{R}^+$ be the relocation cost from moving the temporary facility from location k to location i at time period t . To account for maintenance costs, we can simply add the term $-f_i^t y_i^t$ to the objective function. To account for relocation costs, we first introduce variables $v_{kt}^t \in \{0, 1\}$, which are equal to 1 if the temporary facility moves from location k to location i at the beginning of time period t , and 0 otherwise. Then, we append constraints $v_{ki}^t = y_k^{t-1} y_i^t$, which can be linearized with McCormick envelopes, to properly control variables v_{kt}^t , and add the term $-g_{ki}^t v_{ki}^t$ to the objective function. Our solution methods, further presented in Section 5, can also be adapted in a straightforward manner to account for these additional extensions.

4 Computational Complexity

In this section, we study whether the CDSFLP-CCD is theoretically tractable. We characterize special cases that are NP-hard with or without approximation guarantees, as well as special cases solvable in polynomial time. For the sake of brevity, we present the proofs of Theorems 2–5 in Appendix B. Before proceeding, let us formally define the decision version of the CDSFLP-CCD.

Decision version of the CDSFLP-CCD:

INSTANCE: Finite sets $\mathcal{T} = \{1, \dots, T\}$, $\mathcal{I} = \{1, \dots, I\}$ and $\mathcal{J} = \{1, \dots, J\}$, positive rational numbers $\{r_i\}_{i \in \mathcal{I}}$ and $\{d_j^t\}_{j \in \mathcal{J}, t \in \mathcal{T}}$, binary values $\{a_{ij}\}_{i \in \mathcal{I}, j \in \mathcal{J}}$ and a positive rational number Π .

QUESTION: Is there a feasible location sequence \mathbf{y} with a total reward of at least Π ?

Lifting the assumption that customer demand is cumulative yields the Dynamic Single Facility Location Problem (DSFLP), which can be written as follows based on the formulation of

Gunawardane (1982):

$$\max_{\mathbf{y}} \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} r_i a_{ij} d_j^t y_i^t \quad (3a)$$

$$\text{s.t.}: \sum_{i \in \mathcal{I}} y_i^t \leq 1 \quad \forall t \in \mathcal{T} \quad (3b)$$

$$y_i^t \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}. \quad (3c)$$

Objective Function (3a) maximizes the total reward, Constraints (3b) guarantee that the provider installs at most one single facility per time period t , and Constraints (3c) define feasible variable domains. We can easily build the optimal location sequence \mathbf{y} for the DSFLP by selecting the location with largest reward in each time period. In other words, the DSFLP is a computationally easy problem, as stated by Proposition 2.

Proposition 2 *The DSFLP is polynomially solvable.*

In this sense, we study the theoretical intractability of the CDSFLP-CCD to understand which problem characteristics, when interacting with cumulative customer demand, may render it NP-hard. We show that the CDSFLP-CCD is NP-hard through a reduction from the Set Packing Problem (SPP), which is known to be NP-hard (Karp, 1972) and cannot be approximated within a constant factor (Hazan et al., 2006).

Theorem 2 *The decision version of the CDSFLP-CCD is NP-complete, and the CDSFLP-CCD cannot be approximated within a factor $T^{1-\alpha}$ for any $\alpha > 0$, unless $P = NP$.*

The proof of Theorem 2 heavily relies on preference rules and location rewards by assuming flexible customers and different rewards. Note, however, that some instances may have loyal customers or identical rewards, and the CDSFLP-CCD may become theoretically tractable for these special cases. In this sense, we first investigate the special case with identical rewards, and show that it remains NP-hard through a reduction from the Satisfiability Problem with exactly three variables per clause (3SAT), which is known to be strongly NP-hard (Karp, 1972; Garey and Johnson, 1979).

Theorem 3 *The decision version of the CDSFLP-CCD with identical rewards is NP-complete, and the CDSFLP-CCD is strongly NP-hard.*

Moreover, we can show that the CDSFLP-CCD with identical rewards is approximable through a greedy algorithm that builds a location sequence in reverse order. More specifically, the t -th iteration chooses the best location (*i.e.*, the one that provides the highest marginal contribution) at the $(T-t)$ -th position of the sequence while assuming that no locations have been installed from time period 1 to $(T-t)-1$. This algorithm is an heuristic for general instances of our problem (*i.e.*, not necessarily with identical rewards). We refer to this algorithm as the Backward Greedy Heuristic, and present its pseudocode in Algorithm 1. Since there might be multiple locations with the same marginal contribution, the function *tie_breaker*(\mathcal{K}) takes as input a subset of locations \mathcal{K} and break ties by returning the location with the smallest index.

Algorithm 1 Backward Greedy Heuristic

Require: \mathcal{I} , $\mathcal{T} = \{1, \dots, T\}$, π $\mathbf{y} \leftarrow \mathbf{0}$ **for all** $t = T, \dots, 1$ **do** **for all** $i \in \mathcal{I}$ **do** $y_i^t \leftarrow 1$ $\Pi_i \leftarrow \pi(\mathbf{y})$ $y_i^t \leftarrow 0$ **end for** $\mathcal{K} \leftarrow \arg \max_{i \in \mathcal{I}} \{\Pi_i\}$ $k \leftarrow \text{tie_breaker}(\mathcal{K})$ $y_k^t \leftarrow 1$ **end for****return** Location sequence \mathbf{y} .

Theorem 4 *Algorithm 1 is a 2-approximation algorithm for the CDSFLP-CCD with identical rewards.*

Theorems 2, 3, and 4 show that having different rewards seems to *strengthen* the NP-hardness of the CDSFLP-CCD (*i.e.*, it turns a problem with potential approximation guarantees into a problem without them). We now turn to the special case with loyal customers. Surprisingly, we are able to show that this special case is solvable in polynomial time without further assumptions about location rewards.

Theorem 5 *The CDSFLP-CCD with loyal customers is polynomially solvable.*

Theorem 5, along with Theorem 3, implies that cumulative customer demand by itself does not generate the NP-hardness, but rather its intrinsic interaction with preference rules over the planning horizon.

5 Solution Methods

We now propose several solutions methods for the DSFLP-CCD. We first propose an exact method based on a Benders decomposition (Benders, 1962; Rahmaniani et al., 2017) of the DI Formulation in Section 5.1, along with an analytical procedure to solve the associated dual subproblems. We then provide two myopic heuristics attempting to derive reasonable solutions when the provider ignores (or, equivalently, is unaware of) cumulative customer demand in Section 5.2.

5.1 Exact Benders Decomposition

The CDSFLP-CCD may be solved exactly by providing one of the formulations discussed in Section 3 to off-the-shelf solvers. However, from a practical point of view, each formulation has particular drawbacks. The SI Formulation needs to be linearized through big- M constraints and has, therefore, a weak continuous relaxation. On the other hand, the number of variables x_{ij}^{tt} in the DI Formulation grows quadratically with the size of the planning horizon T , thus requiring more memory to explore a likely larger number of nodes in the branch-and-bound tree until finding an optimal solution and proving its optimality.

We can, however, overcome the challenge faced by the DI Formulation by projecting out variables $x_{ij}^{\ell t}$ through an exact Benders decomposition. The DI Formulation can be rewritten as follows:

$$\max_{\mathbf{y}} \sum_{j \in \mathcal{J}} w_j(\mathbf{y}) \quad (4a)$$

$$\text{s.t.}: \sum_{i \in \mathcal{I}} y_i^t \leq 1 \quad \forall t \in \mathcal{T} \quad (4b)$$

$$y_i^t \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}, \quad (4c)$$

where $w_j(\mathbf{y})$ is the optimal value function for the subproblem of customer j for a location sequence \mathbf{y} .

The primal subproblem $w_j^P(\mathbf{y})$ of customer j can be written as follows:

$$w_j^P(\mathbf{y}) : \max_{\mathbf{x}} \sum_{t \in \mathcal{T}} \sum_{\substack{\ell \in \mathcal{T}^S: \\ \ell < t}} \sum_{i \in \mathcal{I}} r_i D_j^{\ell t} x_i^{\ell t} \quad (5a)$$

$$\text{s.t.}: \sum_{\substack{\ell \in \mathcal{T}^S: \\ \ell < t}} x_i^{\ell t} = a_{ij} y_i^t \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T} \quad (5b)$$

$$\sum_{\substack{s \in \mathcal{T}^S: \\ s < t}} \sum_{i \in \mathcal{I}} x_i^{st} = \sum_{\substack{s \in \mathcal{T}^F: \\ s > t}} \sum_{i \in \mathcal{I}} x_i^{ts} \quad \forall t \in \mathcal{T} \quad (5c)$$

$$\sum_{s \in \mathcal{T}^F} \sum_{i \in \mathcal{I}} x_i^{0s} = 1 \quad (5d)$$

$$x_i^{\ell t} \in \mathbb{R}^+ \quad \forall i \in \mathcal{I}, \forall \ell \in \mathcal{T}^S, \forall t \in \mathcal{T}^F : \ell < t, \quad (5e)$$

where we omitted the index j of primal variables $x_{ij}^{\ell t}$ for the sake of simplicity. The associated dual subproblem $w_j^D(\mathbf{y})$ can be written as follows:

$$w_j^D(\mathbf{y}) : \min_{\mathbf{p}, \mathbf{q}} \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} a_{ij} y_i^t p_i^t + q^0 \quad (6a)$$

$$\text{s.t.}: p_i^t + q^\ell - q^t \geq r_i D_j^{\ell t} \quad \forall i \in \mathcal{I}, \forall \ell \in \mathcal{T}^S, \forall t \in \mathcal{T} : \ell < t \quad (6b)$$

$$q^\ell \geq 0 \quad \forall \ell \in \mathcal{T}^S \quad (6c)$$

$$p_i^t \in \mathbb{R} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T} \quad (6d)$$

$$q^t \in \mathbb{R} \quad \forall t \in \mathcal{T}^S, \quad (6e)$$

where dual variables p_i^t are related to Constraints (5b) and dual variables q^t are related to Constraints (5c)–(5d). Finally, the restricted master problem (*i.e.*, with some optimality cuts) can be written as follows:

$$\max_{\mathbf{w}, \mathbf{y}} \sum_{j \in \mathcal{J}} w_j \quad (7a)$$

$$\text{s.t.}: \sum_{i \in \mathcal{I}} y_i^t \leq 1 \quad \forall t \in \mathcal{T} \quad (7b)$$

$$w_j \leq \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} a_{ij} p_i^{t*} y_i^t + q^{0*} \quad \forall j \in \mathcal{J}, \forall (\mathbf{p}^*, \mathbf{q}^*) \in \mathcal{O}_j \quad (7c)$$

$$w_j \in \mathbb{R}^+ \quad \forall j \in \mathcal{J} \quad (7d)$$

$$y_i^t \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}, \quad (7e)$$

where $w_j \in \mathbb{R}^+$ estimates of the optimal value function for the subproblem of customer j based on the set of optimality cuts \mathcal{O}_j already generated for customer j . More specifically, whenever we solve the dual subproblem $w_j^D(\mathbf{y})$ for a location sequence \mathbf{y} , we add its optimal solution $(\mathbf{p}^*, \mathbf{q}^*)$ to set \mathcal{O}_j .

The primal subproblem $w_j^P(\mathbf{y})$ has a feasible region similar to network flow problems, which are known to suffer from degeneracy. Such a characteristic induces multiple optimal solutions for the dual subproblem $w_j^D(\mathbf{y})$ (*i.e.*, multiple optimality cuts for the same location sequence \mathbf{y}). In this sense, the optimality cuts given by simply obtaining some optimal solution of the dual subproblem $w_j^D(\mathbf{y})$ may be shallow or lack structure among themselves (*e.g.*, Magnanti and Wong, 1981). In addition, preliminary experiments have shown that solving J subproblems might become a bottleneck whenever the number of customers is large, which has also been observed in the literature (*e.g.*, Cordeau et al., 2019). To address these challenges, we devise an analytical procedure to compute optimality cuts, presented in Algorithm 2.

Algorithm 2 Analytical Solution of Subproblem $w_j^D(\mathbf{y})$

Require: $\mathcal{I}, \mathcal{T} = \{1, \dots, T\}, \mathbf{y}, \mathbf{x}^*$
for all $v = 1, 0$ **do**
 for all $\ell = T, \dots, 1 \mid \sum_{i \in \mathcal{I}} a_{ij} y_i^\ell = v$ **do**
 $q^{\ell*} \leftarrow \max_{\substack{i \in \mathcal{I}, s \in \mathcal{T}^s, t \in \mathcal{T}: \\ s < t, x_i^{st*} = 1 \\ \ell < t, \ell \neq s}} \{r_i(D_j^{\ell t} - D_j^{st}) + q^{s*}, 0\}$
 end for
end for
for all $t \in \mathcal{T}$ **do**
 for all $i \in \mathcal{I}$ **do**
 $p_i^{t*} \leftarrow \max_{\ell \in \mathcal{T}: \ell < t} \{r_i D_j^{\ell t} - q^{\ell*} + q^{t*}\}$
 end for
end for
return Optimality cut $(\mathbf{p}^*, \mathbf{q}^*)$.

Algorithm 2 is based on two key ideas. The first one is that we can project out variables p_i^t from the dual subproblem $w^D(\mathbf{y})$, and compute their values based on variables q^t . The second one is that we can compute feasible values for variables q^t based on the optimal solution \mathbf{x}^* of the primal subproblem $w^P(\mathbf{y})$. Then, by strong duality, we are able to show that these feasible values for variables q^t and, consequently, variables p_i^t are indeed optimal. Given a location sequence \mathbf{y} , we can compute the optimal solution \mathbf{x}^* of the primal problem $w^P(\mathbf{y})$ in polynomial time by inspection, as stated by Proposition 1. Then, we call Algorithm 2 to first compute feasible values for variables q^t , and then feasible values for variables p_i^t . When computing feasible values for variables q^t , it is important to first compute them for time periods where customer j was captured by the facility ($v = 1$), then for time periods where customer j was not captured by the facility ($v = 0$). We remark that the maximization problems within Algorithm 2 can be solved by inspection.

Theorem 6 *The optimal solution $(\mathbf{p}^*, \mathbf{q}^*)$ of the dual subproblem $w_j^D(\mathbf{y})$ can be found analytically through Algorithm 2, where \mathbf{x}^* is the optimal solution of the primal subproblem $w_j^P(\mathbf{y})$.*

Proof. See Appendix B.6.

We do not claim that this analytical procedure produces tighter cuts, but we ensure that generated cuts share the same structure. We implement the exact Benders decomposition in a branch-and-cut fashion through callbacks, further referred to as branch-and-Benders-cut (Cordeau et al., 2019), where we add optimality cuts whenever the solver finds a feasible location sequence (*i.e.*, binary values for variables y_i^t).

5.2 Myopic Heuristic Methods

The CDSFLP-CCD may be faced by providers that explicitly or implicitly ignore cumulative customer demand when devising a location sequence. For example, the provider might assume that unmet demand vanishes and employ the DSFLP formulation presented in Section 4 to obtain a location sequence. In this sense, we present two myopic heuristics to derive what seem to be natural solutions when cumulative demand is overlooked. In the computational experiments, we employ these heuristics to evaluate the economical benefit of modelling cumulative customer demand.

DSFLP-based Heuristic. Proposition 2 states that the DSFLP can be solved in polynomial time. Given the simplicity of this procedure, it may be employed to obtain a location sequence \mathbf{y} that completely ignores demand accumulation, and evaluate how it performs by computing the total reward $\pi(\mathbf{y})$. This heuristic is likely to output solutions where the temporary facility remains in the same location or visits the same subset of locations throughout the planning horizon.

Forward Greedy Heuristic. This heuristic follows a myopic approach of perceiving customer demand at each time period and then choosing the best location (*i.e.*, the one that provides the highest marginal contribution) accordingly. We highlight that this heuristic does not ignore demand accumulation, but rather neglects future effects of current location decisions. Although the Backward Greedy Heuristic presented in Section 4 is similar to the Forward Greedy Heuristic in nature, we cannot trivially extend the approximation guarantees of the former to the latter (see Theorem 4). Algorithm 3 presents the pseudocode of this heuristic.

Algorithm 3 Forward Greedy Heuristic

Require: \mathcal{I} , $\mathcal{T} = \{1, \dots, T\}$, π
 $\mathbf{y} \leftarrow \mathbf{0}$
for all $t = 1, \dots, T$ **do**
 for all $i \in \mathcal{I}$ **do**
 $y_i^t \leftarrow 1$
 $\Pi_i \leftarrow \pi(\mathbf{y})$
 $y_i^t \leftarrow 0$
 end for
 $\mathcal{K} \leftarrow \arg \max_{i \in \mathcal{I}} \{\Pi_i\}$
 $k \leftarrow \text{tie_breaker}(\mathcal{K})$
 $y_k^t \leftarrow 1$
end for
return Location sequence \mathbf{y} .

6 Computational Experiments

In this section, we study the empirical performance of the proposed exact and heuristic methods. In Section 6.1, we describe the experimental setup, as well as the instance generation procedure. Theorem 1 guarantees that the DI Formulation provides a tighter continuous relaxation than the SI Formulation, which should help off-the-shelf solvers to find integer solutions and prove optimality faster. However, this is not always the case nowadays due to other built-in techniques that might impact the branch-and-bound tree in an unexpected (or randomized) manner. In Section 6.2, we first evaluate how well Gurobi performs in terms of solution times when solving the SI and DI Formulations. Then, in Section 6.3, we analyze the improvement of the DI Formulation with the exact Benders decomposition. In particular, we assess whether having an analytical procedure to compute optimality cuts has a relevant impact on the effectiveness of the Benders decomposition. Lastly, in Section 6.4, we investigate the performance of the heuristic solutions methods. More specifically, we evaluate whether the heuristics provide sufficiently high-quality solutions to quantify the economical benefit of explicitly modelling cumulative customer demand.

6.1 Experimental Setup

We implemented the majority of our solution methods in Python (version 3.8), except for our analytical procedure used in the Benders decomposition written in C, and solved the mixed-integer programs with Gurobi (version 9.5). We employ C to ensure a fair comparison with Gurobi, which also runs on C, in terms of solution times when solving dual subproblems. Each solution method had a time limit of 5 hours in total and was limited to a single thread to avoid bias related to computational resources. All jobs were processed on the Beluga cluster of the Digital Research Alliance of Canada, where each node has 30GB of RAM and 2 CPUs (Intel Gold 6148 Skylake, 2.4 GHz).

We further refer to our solution methods as follows. SIF and DIF represent solving the SI Formulation (after linearization) and the DI Formulation, respectively, with Gurobi. Then, BSD and BSA refer to the branch-and-Benders-cut implementations, where the former solves dual subproblems with Gurobi and the latter solves dual subproblems with the analytical procedure. Finally, DBH, FGH, and BGH describe the DSFLP-based Heuristic, the Forward Greedy Heuristic, and the Backward Greedy Heuristics, respectively. As a sanity check, we also evaluate a randomly generated solution denoted as RND.

Since the CDSFLP-CCD is a novel problem, benchmark instances are unavailable. We therefore generate synthetic instances inspired by other papers in the literature (*e.g.*, Zarandi et al., 2013; Marín et al., 2018). We consider ten different seed values for random parameters, fix $T = 10$ time periods, and generate other parameters as follows. We consider $I \in \{50, 100\}$ candidate locations and fix $J = I$ targeted customers. We consider $P \in \{\frac{1}{2}, 2\}$ to generate preference rules. Each customer $j \in \mathcal{J}$ samples uniformly at most $\lceil \frac{PI}{T} \rceil$ locations to build a choice set C_j and set

$$a_{ij} = \begin{cases} 1, & \text{if } i \in C_j \text{ or } i = j \\ 0, & \text{otherwise} \end{cases}, \forall i \in \mathcal{I}, \text{ where } P = \frac{1}{2} \text{ (resp., } P = 2) \text{ generates instances where}$$

customers have a small (resp., large) choice set, and are likely to be visited at most once (resp., more than once) throughout the planning horizon. Although most works in the literature generate preference rules based on the geographical site of locations and customers (*e.g.*, customers are willing to attend locations within some radius from them), we employ a random choice set to create instances where personal preferences are not strictly tied to geographical distance (*e.g.*, customers may prefer to attend locations that are farther from their neighbourhood if they are easily accessible by public transportation or have a better ambience). We consider identical ($r_i = I, \forall i \in \mathcal{I}$) and

different ($r_i = \lceil \frac{I}{\sum_{j \in \mathcal{J}} a_{ij}} \rceil, \forall i \in \mathcal{I}$) rewards per unit of captured demand. Intuitively, the former describes applications where the reward is independent of location, whereas the latter describes applications where popular locations tend to have larger costs and, naturally, smaller rewards. We consider two types of demand functions over time: constant ($d_j^t = D, \forall j \in \mathcal{J}, \forall t \in \mathcal{T}$) and seasonal ($d_j^t = \lceil \frac{D}{2} \cos t + \frac{D}{2} \rceil, \forall j \in \mathcal{J}, \forall t \in \mathcal{T}$), where $D = 20$ is a constant.

This instance generation process yields $2^4 = 16$ instances per seed (*i.e.*, $10 \times 16 = 160$ instances in total). We refer to this benchmark as *homogeneous* because customers have the same spawning demand function over time with amplitude $D = 20$. We also generate an *heterogeneous* benchmark by sampling uniformly an amplitude $D_j \sim \{10, 15, 20, 25, 30\}$ for each customer j . This benchmark adds another $2^4 = 16$ instances per seed, (*i.e.*, $10 \times 16 = 160$ in total). The complete benchmark therefore contains $2 \times 160 = 320$ instances.

6.2 Comparison between the SI and DI Formulations

We first compare SIF and DIF based on instances that were solved to optimality by both formulations. We compute the integrality gap as $\frac{\Pi' - \Pi^*}{\Pi'}$, where Π^* is the optimal objective value of some mixed-integer formulation and Π' is the optimal objective value of its continuous relaxation. In Table 1, columns “int. gap (%)” and “time (min)” present the average integrality gap and the average solution time of SIF and DIF, as well as their standard deviations, for instances solved to optimality by both formulations. The column “% instances considered” presents the percentage of instances with a certain attribute that were considered to compute the average values.

Table 1: Average integrality gaps and solution times of SIF and DIF, as well as their standard deviations, for instances solved to optimality by both formulations.

Instance attributes	% instances considered	SIF		DIF	
		int. gap (%)	time (min)	int. gap (%)	time (min)
Complete benchmark	74.69	7.14 ± 4.83	17.36 ± 37.83	1.75 ± 1.43	7.78 ± 29.15
50 locations, customers	98.75	8.69 ± 4.84	23.42 ± 42.47	1.94 ± 1.49	10.66 ± 33.65
100 locations, customers	50.62	4.10 ± 3.07	5.55 ± 22.45	1.37 ± 1.23	2.17 ± 16.03
Small choice sets	100.00	4.48 ± 2.92	1.91 ± 2.28	1.11 ± 1.08	0.22 ± 0.35
Large choice sets	49.38	12.51 ± 3.19	48.64 ± 53.63	3.04 ± 1.16	23.09 ± 47.30
Identical rewards	75.62	8.54 ± 3.31	6.77 ± 19.82	2.02 ± 1.05	1.79 ± 13.12
Different rewards	73.75	5.69 ± 5.67	28.22 ± 47.69	1.47 ± 1.70	13.92 ± 38.43
Constant demand	75.00	8.31 ± 5.42	17.51 ± 41.45	1.65 ± 1.34	5.64 ± 20.91
Seasonal demand	74.38	5.95 ± 3.84	17.21 ± 33.97	1.85 ± 1.51	9.93 ± 35.55
Identical amplitudes	73.75	6.99 ± 5.15	22.52 ± 46.13	1.94 ± 1.46	13.08 ± 38.58
Sampled amplitudes	75.62	7.28 ± 4.52	12.33 ± 26.69	1.55 ± 1.38	2.61 ± 13.39

Overall, SIF presents an average integrality gap approximately four times larger than DIF, with a similar trend for varying instance attributes. These lower integrality gaps seem to play an important role when solving the problems, as DIF takes less than half the average solution time of SIF to solve the same subset of instances to optimality. Table 1 also provides some intuition on which instance attributes induce programs that are considerably harder to solve. We can see that large choice sets and different rewards imply higher solution times for both formulations, which relate to our theoretical results in Section 4. Small choice sets induce instances with some loyal customers, which are closer to the special case solvable in polynomial time, and setting different rewards seem to be key in strengthening the complexity of the problem. In addition, we can

see that identical amplitudes also imply higher solution times. Intuitively, identical amplitudes lead to instances with similar customers from a demand perspective, and this seems to hinder the effectiveness of branching on variables y_i^t in reducing the upper bound. Indeed, a closer look into a subset of instances with identical amplitudes shows that Gurobi spends approximately 70% of the solution time, on average, solely closing the gap after having found the optimal solution.

We now compare SIF and DIF based on instances that were not solved to optimality by at least one of them. In Table 2, columns “# opt.” and “opt. gap (%)” present, respectively, the number of instances solved to optimality and the average optimality gap (and its standard deviation) of SIF and DIF. We consider here the optimality gap reported by Gurobi at the end of the time limit.

Table 2: Number of instances solved to optimality, average optimality gaps and their standard deviations, of SIF and DIF, for instances not solved to optimality by at least one of the formulations.

Instance attributes	% instances considered	SIF		DIF	
		# opt.	opt. gap (%)	# opt.	opt. gap (%)
Complete benchmark	25.31	2	7.57 ± 4.07	14	3.96 ± 3.10
50 locations, customers	1.25	2	0.00 ± 0.01	0	0.80 ± 0.49
100 locations, customers	49.38	0	7.76 ± 3.94	14	4.04 ± 3.10
Small choice sets	0.00	–	–	–	–
Large choice sets	50.62	2	7.57 ± 4.07	14	3.96 ± 3.10
Identical rewards	24.38	0	4.09 ± 1.56	14	1.39 ± 1.35
Different rewards	26.25	2	10.81 ± 2.81	0	6.34 ± 2.23
Constant demand	25.00	1	7.00 ± 4.52	11	3.45 ± 3.16
Seasonal demand	25.62	1	8.13 ± 3.55	3	4.45 ± 2.99
Identical amplitudes	26.25	2	7.88 ± 4.27	6	4.62 ± 3.38
Sampled amplitudes	24.38	0	7.24 ± 3.88	8	3.24 ± 2.61

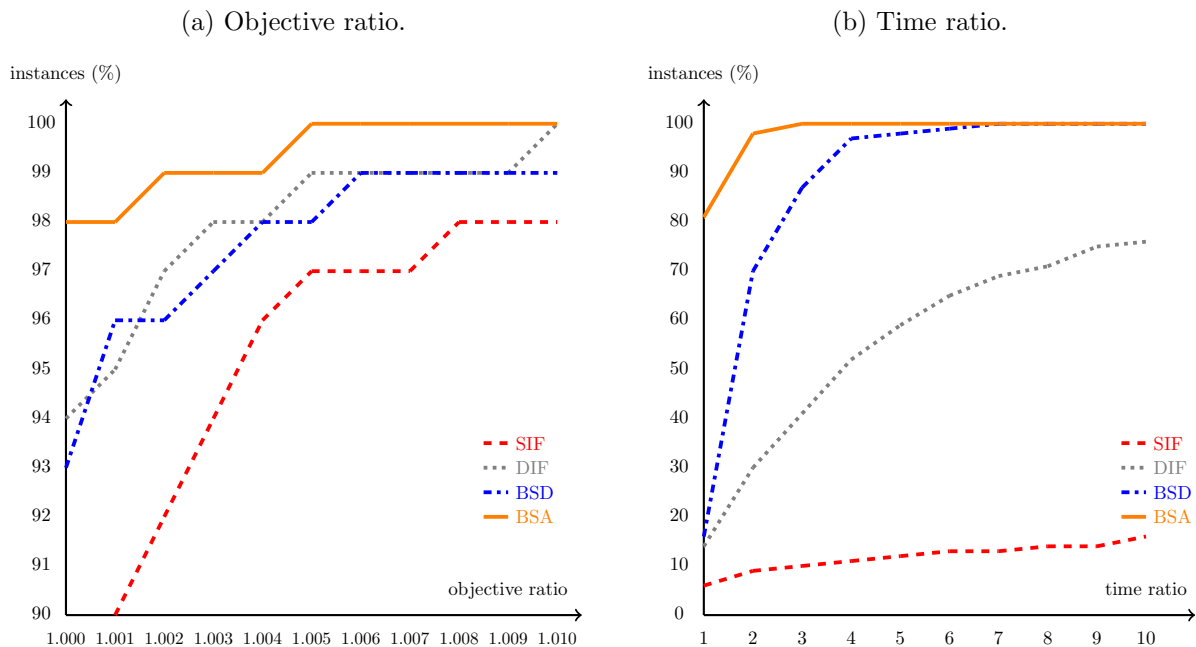
DIF outputs the optimal solution for a larger number of instances than SIF (14 versus 2 instances), and proves a smaller average optimality gap (3.96% versus 7.57%) within the same time limit. This trend remains the same for varying instance attributes, except for instances with 50 locations. A closer look into these two instances, where DIF has a larger average optimality gap than SIF, shows that it is a result of the previously explained behaviour induced by identical amplitudes. Nevertheless, on average, DIF outperforms SIF from a theoretical and practical point of view, as it finds the optimal solution and proves optimality faster, while guaranteeing a lower optimality gap within the same time limit.

6.3 Performance of the Exact Benders Decomposition

We now focus on the performance of BSD and BSA, where dual subproblems are solved by Gurobi and the analytical procedure, respectively. First, we compare BSD and BSA with SIF and DIF in terms of solution quality and computing times. To this end, we compute the objective ratio of each exact method as $\frac{\Pi^b}{\Pi'}$, where Π^b is the highest objective value found by SIF, DIF, BSD, and BSA, and Π' is the objective value obtained by the exact method at hand. Similarly, we compute the time ratio of each exact method as $\frac{\Delta'}{\Delta^b}$, where Δ^b is the lowest solution time among SIF, DIF, BSD, and BSA, and Δ' is the solution time taken by the exact method at hand. Small (resp., large) objective ratios indicate that the exact method at hand finds a solution with an objective

value closer (resp., farther) to the best objective value. Similarly, small (resp., large) time ratios indicate that the exact method at hand has a solution time closer (resp., farther) to the fastest solution time. Figure 3 presents the performance profile for SIF, DIF, BSD, and BSA in terms of solution quality and computing times, where the y axis presents the percentage of instances with an ratio smaller than or equal to the reference value on the x axis for each exact method.

Figure 3: Performance profiles for objective values and computing times of SIF, DIF, BSD, and BSA, where the y axis presents the percentage of instances with an objective ratio or time ratio, respectively, smaller or equal to the reference value on the x axis for each exact method.



In terms of solution quality, Figure 3a highlights BSA and SIF as the best and worst exact methods, respectively, while DIF and BSD have a similar performance. This result indicates that applying the Benders decomposition by itself does not guarantee the discovery of better solutions within the same time limit. Solving the dual problem analytically within BSA does, however, improve the solution quality. These conclusions are in line with an analysis of computing times, as Figure 3b shows that BSA is also the fastest solution time among exact methods.

We keep DIF as the baseline, and now analyze BSD and BSA in terms of solution times and optimality gaps. First focusing on instances solved to optimality by all the three exact methods (*i.e.*, DIF, BSD, and BSA), similarly to Table 1, Table 3 presents average solution times for each method. On average, BSD and BSA are approximately 16 and 30 times faster than DIF, respectively. This advantage seems more dramatic for some instance attributes, namely 100 locations and identical rewards, where BSA is approximately 130 times faster than DIF. Larger number of locations induce a quadratic increase on the number of variables x_{ij}^{lt} in DIF since $J = I$ in our benchmark, which is avoided by the structure of the Benders decomposition. It seems that optimality cuts are particularly effective in approximating subproblems for identical rewards, thus circumventing the major drawback observed in DIF when it comes to closing the gap. We also observe, once again, that employing an analytical procedure to solve dual subproblems has a impact on solution times, as BSA is 1.8 times faster than BSD on average and remains faster for varying instance attributes.

Table 3: Average solution times and their standard deviations of DIF, BSD, and BSA, for instances solved to optimality by all three exact methods.

Instance attributes	% instances considered	DIF	BSD	BSA
		time (min)	time (min)	time (min)
Complete benchmark	79.06	20.38 ± 60.34	1.24 ± 4.60	0.69 ± 1.94
50 locations, customers	98.75	10.66 ± 33.65	1.77 ± 5.75	0.93 ± 2.40
100 locations, customers	59.38	36.54 ± 86.29	0.36 ± 0.54	0.28 ± 0.46
Small choice sets	100.00	0.22 ± 0.35	0.10 ± 0.09	0.07 ± 0.08
Large choice sets	58.12	55.05 ± 89.72	3.20 ± 7.20	1.74 ± 2.92
Identical rewards	84.38	26.01 ± 74.08	0.26 ± 0.47	0.20 ± 0.40
Different rewards	73.75	13.92 ± 38.43	2.36 ± 6.56	1.24 ± 2.72
Constant demand	81.88	24.21 ± 66.06	1.49 ± 5.85	0.71 ± 1.99
Seasonal demand	76.25	16.26 ± 53.48	0.97 ± 2.70	0.66 ± 1.90
Identical amplitudes	77.50	24.71 ± 64.55	1.93 ± 6.40	1.03 ± 2.65
Sampled amplitudes	80.62	16.21 ± 55.92	0.58 ± 1.23	0.36 ± 0.70

We next analyze how these three exact methods perform for instances not solved to optimality by at least one of them. Similarly to Table 2, Table 4 presents the number of instances solved to optimality and optimality gaps of DIF, BSD and BSA. Among instances not solved to optimality by at least one of these methods, BSA outputs the optimal solution for a larger number of instances than BSD (47 versus 36 instances), and proves a smaller optimality gap (0.91% versus 1.86%) within the same time limit. Therefore, BSA also outperforms BSD when it comes to finding better solutions within the same time limit.

Table 4: Number of instances solved to optimality, average optimality gaps and their standard deviations, of DIF, BSD, and BSA, for instances not solved to optimality by at least one of these three exact methods.

Instance attributes	% instances considered	DIF		BSD		BSA	
		# opt.	opt. gap (%)	# opt.	opt. gap (%)	# opt.	opt. gap (%)
Complete benchmark	20.94	0	4.78 ± 2.76	36	1.86 ± 2.19	47	0.91 ± 1.63
50 locations, customers	1.25	0	0.80 ± 0.49	2	0.01 ± 0.00	2	0.01 ± 0.00
100 locations, customers	40.62	0	4.90 ± 2.71	34	1.92 ± 2.20	45	0.94 ± 1.64
Small choice sets	0.00	0	–	0	–	0	–
Large choice sets	41.88	0	4.78 ± 2.76	36	1.86 ± 2.19	47	0.91 ± 1.63
Identical rewards	15.62	0	2.17 ± 1.07	25	0.01 ± 0.00	25	0.01 ± 0.00
Different rewards	26.25	0	6.34 ± 2.23	11	2.97 ± 2.10	22	1.45 ± 1.86
Constant demand	18.12	0	4.76 ± 2.74	12	2.57 ± 2.37	16	1.35 ± 1.86
Seasonal demand	23.75	0	4.80 ± 2.82	24	1.32 ± 1.91	31	0.58 ± 1.35
Identical amplitudes	22.50	0	5.39 ± 3.03	16	2.60 ± 2.43	19	1.60 ± 1.95
Sampled amplitudes	19.38	0	4.07 ± 2.27	20	1.01 ± 1.52	28	0.11 ± 0.41

Finally, we evaluate the effectiveness of the optimality cuts generated within BSD and BSA. To this end, we analyze the number of callback calls, which represents the number of location sequences analyzed by Gurobi in the branch-and-bound tree. In Table 5, columns “# callbacks”

and “% time in callbacks” present the average number of callbacks and the average percentage of solution time spent in callbacks within BSD and BSA, for instances solved to optimality by both branch-and-Benders-cut implementations.

First, we highlight that BSA has, on average, less callback calls than the BSD, and that for varying instance attributes. This indicates that employing the analytical procedure to solve dual subproblems provides a more effective set of optimality cuts than doing so with Gurobi. We also note that the same instance attributes that generate harder mixed-integer programs require more optimality cuts to be solved, namely different rewards and large choice sets, which is connected to our theoretical results in Section 4. Second, we highlight that BSA spends, on average, less time than BSD solving dual subproblems, no matter the instance attribute. As a result, the speed of the analytical procedure leaves more time for Gurobi to explore the branch-and-bound tree within the same limit.

Table 5: Average number of callback calls and percentage of solution time spent in callbacks within BSD and BSA, as well as their standard deviations, for instances solved to optimality by both branch-and-Benders-cut implementations.

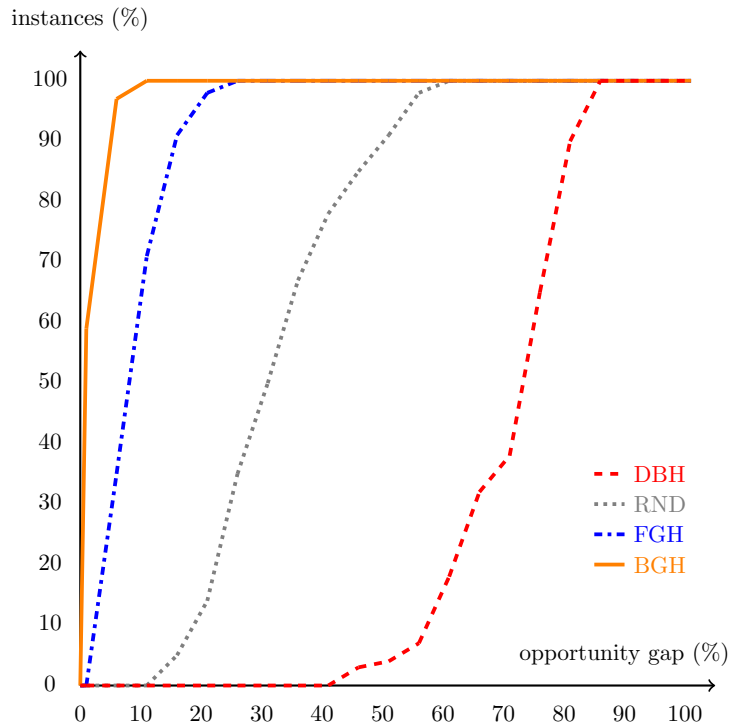
Instance attributes	% instances considered	BSD		BSA	
		# callbacks	% time in callbacks	# callbacks	% time in callbacks
Complete benchmark	90.31	4035.99	36.23	3487.89	10.68
50 locations, customers	100.00	3848.75	37.14	3417.50	11.19
100 locations, customers	80.62	4268.22	35.11	3575.19	10.04
Small choice sets	100.00	3039.38	50.36	2271.88	15.22
Large choice sets	80.62	5272.09	18.72	4996.12	5.03
Identical rewards	100.00	3223.75	37.19	2953.75	9.82
Different rewards	80.62	5043.41	35.05	4150.39	11.74
Constant demand	89.38	3884.62	39.47	3359.44	12.43
Seasonal demand	91.25	4184.25	33.07	3613.70	8.96
Identical amplitudes	87.50	3867.14	29.85	3304.29	6.91
Sampled amplitudes	93.12	4194.63	42.23	3660.40	14.22

6.4 Performance of the Myopic Heuristic Methods

We now evaluate the performance of the heuristics for 300 instances with a known optimal solution. To this end, we define the opportunity gap for each heuristic as $\frac{\Pi^* - \Pi'}{\Pi^*}$, where Π^* is the optimal objective value obtained through some exact method and Π' is the objective value of the heuristic at hand. Intuitively, large (resp., small) opportunity gaps indicate that the heuristic finds low-quality (resp., high-quality) location sequences. In particular, small opportunity gaps indicate that the heuristic could be employed whenever the provider cannot apply exact methods (*e.g.*, for large-scale instances or when there are no off-the-shelf solvers available).

Figure 4 presents the performance profile of DBH, RND, FGH and BGH, where the y axis presents the percentage of instances with an opportunity gap smaller or equal to the reference value on the x axis for each heuristic. On the one hand, we can see that DBH has the worst performance in terms of opportunity gap, even worse than RND (*i.e.*, our sanity check). In other words, if the provider devises a location sequence without taking into account cumulative customer demand through the DSFLP, the total reward obtained is worse than selecting locations to visit

Figure 4: Performance profile for opportunity gaps of DBH, RND, FGH and BGH, where the y axis presents the percentage of instances with an opportunity gap smaller or equal to the reference value on the x axis for each heuristic.



randomly at each time period. Recall that DBH tends to visit the same location or the same subset of locations throughout the planning horizon based on the spawning demand, thus not taking advantage from visiting customers with accumulated demand. On the other hand, FGH and BGH perform considerably better than the sanity check, with the former having a clear worse performance in terms of opportunity gaps than the latter in terms of opportunity gap.

We now focus on the performance of the heuristics for varying instance attributes. In Table 6, the column “opp. gap (%)” presents the average opportunity gap (and its standard deviation) of DBH, RND, FGH and BGH, for instances with a known optimal solution.

As expected, DBH has an incredibly high average optimality gap of 70.05%, and that for varying instance attributes. We highlight that FGH has a lower average opportunity gap of 8.52% in comparison to DBH, but still considerably high. In this sense, if the provider decides to act myopically and choose at each time period the location with the highest marginal contribution, there is some considerable lost opportunity in ignoring cumulative customer behaviour. Therefore, cumulative customer demand cannot be ignored, and should be accounted for in the optimization framework to obtain high-quality solutions.

We then highlight that BGH performs surprisingly much better than expected, even under different rewards, having a small average opportunity gap around 1.27%. This result is somewhat expected, since the proof of Theorem 4 is very generous when providing the upper bound of the optimal objective value through the heuristic objective value, and may indicate that tighter approximation guarantees may be achievable. This heuristic also seems to scale well in terms of locations

and customers, as the opportunity gap does not change a lot from 50 to 100 locations. In this sense, if the provider needs to solve large-scale instances where exact methods may not be applicable, this heuristic represents a sufficiently reasonable alternative.

Table 6: Average opportunity gaps and their standard deviations of DBH, RND, FGH and BGH, for instances with a known optimal solution.

Instance attributes	% instances considered	DBH	RND	FGH	BGH
		opp. gap (%)	opp. gap (%)	opp. gap (%)	opp. gap (%)
Complete benchmark	93.75	70.05 ± 9.91	32.60 ± 11.44	8.52 ± 4.84	1.27 ± 1.61
50 locations, customers	100.00	69.48 ± 9.83	34.38 ± 10.19	9.56 ± 5.23	1.33 ± 1.81
100 locations, customers	87.50	70.70 ± 10.00	30.56 ± 12.45	7.33 ± 4.05	1.20 ± 1.34
Small choice sets	100.00	75.20 ± 7.00	37.92 ± 12.30	7.86 ± 2.87	0.78 ± 0.97
Large choice sets	87.50	64.17 ± 9.48	26.52 ± 6.21	9.28 ± 6.31	1.83 ± 1.98
Identical rewards	100.00	66.79 ± 8.44	35.11 ± 13.25	6.88 ± 3.02	0.78 ± 1.02
Different rewards	87.50	73.78 ± 10.18	29.73 ± 8.07	10.40 ± 5.76	1.83 ± 1.95
Constant demand	91.88	72.17 ± 7.47	32.82 ± 11.60	8.53 ± 4.75	1.14 ± 1.49
Seasonal demand	95.62	68.02 ± 11.46	32.39 ± 11.31	8.51 ± 4.94	1.40 ± 1.71
Identical amplitudes	89.38	73.83 ± 7.54	30.45 ± 11.66	7.45 ± 4.96	1.47 ± 1.86
Sampled amplitudes	98.12	66.61 ± 10.56	34.56 ± 10.90	9.50 ± 4.52	1.09 ± 1.32

7 Conclusion

We investigate a novel multi-period deterministic location problem, where the decision maker relocates a single temporary facility over time to capture cumulative customer demand. This paper makes practical, methodological and theoretical contributions. On the practical front, we model demand behaviour that, despite its practical relevance, has not received much attention in the literature. Indeed, in our computational experiments, ignoring cumulative demand has resulted in an average loss of 70% of the optimal reward.

From a methodological perspective, we model this problem as a mixed-integer program and present a reformulation as an integer program, which provides a tighter continuous relaxation. While the latter is solved to optimality in less than half of the solution time as the former, it also allows for a more general customer demand behaviour. We then devise an exact Benders decomposition for our reformulation with an analytical procedure to generate optimality cuts, which is 30 times faster than solving the reformulation directly. Myopic heuristics yield low-quality solutions (at best, within 8% of the optimal solution, on average), highlighting the importance of modelling cumulative demand behaviour.

On the theoretical front, we identify and proof which problem characteristics reduce or increase the computational complexity. We also present a 2-approximate algorithm for the special case with identical rewards at each location. Although this algorithm is a heuristic for the general problem (*i.e.*, not necessarily with identical rewards), it still finds reasonably high-quality solutions for our benchmark (on average, within 2% of the optimal solution), thus being an interesting alternative for large-scale instances.

Given the potential relevance of modelling cumulative demand in other application contexts, we hope that the here provided modelling techniques, solution methods and theoretical insights will be useful to more realistically model and solve such planning problems. Future work includes (*i*) studying our planning problem in a duopoly, where the provider competes over customers with

a competitor that can provide the same service, and (ii) the explicit representation of parameters uncertainty, as, for example, either customer behaviour or demand quantities may be difficult to accurately estimate in advance. These two settings may be faced by multiple real-world applications, and may considerably impact the structure of the optimal location sequence implemented by the service provider.

A Linearization Details

In this appendix, we provide the detailed linearization of the SI Formulation.

We employ additional decision variables $w_{ij}^t \in \mathbb{R}^+$ to store the demand of customer j captured by location i at time period t , and additional parameters $M_j^t \in \mathbb{R}^+$ to represent a sufficiently large constant for each time period t and each customer j . The linearized version of the SI Formulation writes as follows:

$$\max_{\mathbf{b}, \mathbf{c}, \mathbf{w}, \mathbf{y}} \quad \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} r_i w_{ij}^t \quad (8a)$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} y_i^t \leq 1 \quad \forall t \in \mathcal{T} \quad (8b)$$

$$b_j^0 = 0 \quad \forall j \in \mathcal{J} \quad (8c)$$

$$c_j^t = b_j^{t-1} + d_j^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8d)$$

$$b_j^t = c_j^t - \sum_{i \in \mathcal{I}} w_{ij}^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8e)$$

$$w_{ij}^t \leq M_j^t a_{ij} y_i^t \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8f)$$

$$w_{ij}^t \leq c_j^t + M_j^t (1 - a_{ij} y_i^t) \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8g)$$

$$w_{ij}^t \geq -M_j^t a_{ij} y_i^t \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8h)$$

$$w_{ij}^t \geq c_j^t - M_j^t (1 - a_{ij} y_i^t) \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8i)$$

$$b_j^t \in \mathbb{R}^+ \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T}^S \quad (8j)$$

$$c_j^t \in \mathbb{R}^+ \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8k)$$

$$w_{ij}^t \in \mathbb{R}^+ \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (8l)$$

$$y_i^t \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}. \quad (8m)$$

Continuous variables w_{ij}^t linearize the bilinear term $w_{ij}^t y_i^t$ in Objective Function (1b). In this sense, Objective Function (8a) and Constraints (8e) are straightforward adaptations of their counterparts in the SI Formulation, whereas Constraints (8b)–(8c) and (8j)–(8m) remain unchanged. Constraints (8f)–(8i) linearize Constraints (1f). If customer j is captured by location i at time period t (*i.e.*, $a_{ij} y_i^t = 1$), Constraints (8f) and (8h) become nonrestrictive, whereas Constraints (8g) and (8i) become restrictive, thus ensuring $w_{ij}^t = c_j^t$. The exact opposite happens if customer j is not captured by location i at time period t (*i.e.*, $a_{ij} y_i^t = 0$), thus ensuring $w_{ij}^t = 0$. Setting tight values for parameters M_j^t is important to obtain a tighter continuous relaxation, which in turn tends to help off-the-shelf solvers employing branch-and-bound to find optimal solutions and prove optimality faster. In the computational experiments, we set $M_j^t = D_j^{0t}$.

B Mathematical Proofs

In this appendix, we provide the mathematical proofs for theoretical results presented throughout the paper. More specifically, Appendices B.1–B.6 present the proofs for Theorems 1–6, respectively.

Before proceeding, we define some additional notation useful for some proofs presented in this appendix. We recall that a location sequence \mathbf{y} can be written as i_1, \dots, i_T without loss of generality. We first define the set of captured customers by a location i or a subset of locations \mathcal{I}' .

Definition 5 (Captured customer set) *The set of customers captured by location $i \in \mathcal{I}$ is $\mathcal{J}(i) = \{j \in \mathcal{J} \mid a_{ij} = 1\}$. Similarly, the set of customers captured by a subset of locations $\mathcal{I}' \subseteq \mathcal{I}$ is $\mathcal{J}(\mathcal{I}') = \bigcup_{i \in \mathcal{I}'} \mathcal{J}(i)$.*

Then, we define the time period where customer j was lastly captured within a feasible solution i_1, \dots, i_T , which allow us to compute the accumulated demand of said customer at time period t .

Definition 6 (Time of previous capture) *Let i_1, \dots, i_T be a feasible solution of a CDSFLP-CCD instance. The time of previous capture of customer j in the feasible solution i_1, \dots, i_T before time period t is*

$$\tau(j, t \mid i_1, \dots, i_T) = \max\{0, t' \in \mathcal{T} \mid t' < t, j \in \mathcal{J}(i_{t'})\}.$$

Finally, we define the marginal contribution brought by location i_t at time period t to the total reward of the provider, as well as the total reward of the provider for a feasible solution i_1, \dots, i_T , as follows.

Definition 7 (Marginal reward function) *Let i_1, \dots, i_T be a feasible solution of a CDSFLP-CCD instance. The marginal reward function of location i_t at time period t in the feasible solution i_1, \dots, i_T is*

$$\rho(i_t, t \mid i_1, \dots, i_T) = r_{i_t} \sum_{j \in \mathcal{J}(i_t)} \sum_{\substack{s \in \mathcal{T}: \\ s > \tau(j, t \mid i_1, \dots, i_T) \\ s \leq t}} d_j^s.$$

Definition 8 (Total reward function) *Let i_1, \dots, i_T be a feasible solution of a CDSFLP-CCD instance. The total reward function of the feasible solution i_1, \dots, i_T is*

$$\pi(i_1, \dots, i_T) = \sum_{t \in \mathcal{T}} \rho(i_t, t \mid i_1, \dots, i_T).$$

B.1 Proof of Theorem 1

Proof. Let \overline{SI} and \overline{DI} be the linear relaxations of the SI and DI Formulations, respectively. Recall that we employ the linearized version of the SI Formulation presented in Appendix A. We show that the DI Formulation provides a tighter continuous relaxation than the SI Formulation in two steps. First, we prove that the DI Formulation is at least as tight as the SI Formulation by showing that each feasible solution (\mathbf{x}, \mathbf{y}) in \overline{DI} has an equivalent feasible solution $(\mathbf{b}, \mathbf{c}, \mathbf{w}, \mathbf{y})$ in \overline{SI} with the same objective value. Second, we prove that the DI Formulation is strictly tighter than the SI Formulation by providing an example where the optimal objective value of \overline{DI} provides a strictly better bound than the one of \overline{SI} (*i.e.*, closer to the objective value of the optimal integer solution).

First step. We first generate a feasible solution $(\mathbf{b}, \mathbf{c}, \mathbf{w}, \mathbf{y})$ in \overline{SI} from a feasible solution (\mathbf{x}, \mathbf{y}) in \overline{DI} . First, we set variables y_i^t to the same value. Then, we set the remaining variables as follows:

$$w_{ij}^t = \sum_{\substack{\ell \in \mathcal{T}^S: \\ \ell < t}} D_j^{\ell t} x_{ij}^{\ell t} \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (9)$$

$$c_j^t = b_j^{t-1} + d_j^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T} \quad (10)$$

$$b_j^t = c_j^t - \sum_{i \in \mathcal{I}} w_{ij}^t \quad \forall j \in \mathcal{J}, \forall t \in \mathcal{T}. \quad (11)$$

This feasible solution in \overline{SI} has the same objective value as the one in \overline{DI} , as replacing values w_{ij}^t in Objective Function (8a) results in Objective Function (2a). Constraints (8b) are satisfied by definition due to Constraints (2b), and Constraints (8c)–(8e) are satisfied by construction due to equations (10)–(11). We now show that values w_{ij}^t , built with Equations (9), respect Constraints (8f)–(8i) by showing that values $w_j^t = \sum_{i \in \mathcal{I}} w_{ij}^t$ respect Constraints (1f), *i.e.*, the nonlinear version of Constraints (8f)–(8i).

Assume for the sake of contradiction that there is a customer j and a time period t such that Constraints (1f) are not satisfied (*i.e.*, $w_j^t \neq (\sum_{i \in \mathcal{I}} a_{ij} y_i^t) c_j^t$). If $w_j^t < (\sum_{i \in \mathcal{I}} a_{ij} y_i^t) c_j^t$, then $\sum_{i \in \mathcal{I}} \sum_{\ell \in \mathcal{T}^S: \ell < t} D_j^{\ell t} x_{ij}^{\ell t} < (\sum_{i \in \mathcal{I}} a_{ij} y_i^t) c_j^t$ due to Equation (9). We employ Constraints (2c) to rewrite the right-hand side of the previous inequality and obtain $\sum_{i \in \mathcal{I}} \sum_{\ell \in \mathcal{T}^S: \ell < t} D_j^{\ell t} x_{ij}^{\ell t} < \sum_{i \in \mathcal{I}} \sum_{\ell \in \mathcal{T}^S: \ell < t} c_j^t x_{ij}^{\ell t}$, from which we draw $D_j^{\ell t} < c_j^t \forall \ell \in \mathcal{T}^S : \ell < t$. This is an absurd because $c_j^t \leq D_j^{0t}$ by construction due to Equation (10). If $w_j^t > (\sum_{i \in \mathcal{I}} a_{ij} y_i^t) c_j^t$, then $\sum_{i \in \mathcal{I}} \sum_{\ell \in \mathcal{T}^S: \ell < t} D_j^{\ell t} x_{ij}^{\ell t} > (\sum_{i \in \mathcal{I}} a_{ij} y_i^t) c_j^t$ due to Equation (9). We apply the same reasoning to obtain $D_j^{\ell t} > c_j^t \forall \ell \in \mathcal{T}^S : \ell < t$. This is an absurd because $c_j^t \geq d_j^t = D_j^{(t-1)t}$ by construction due to Equation (10). Therefore, Constraints (1f) and, consequently, Constraints (8f)–(8i) are satisfied.

Second step. Consider an instance with three locations $\mathcal{I} = \{1, 2, 3\}$, two customers $\mathcal{J} = \{A, B\}$, and two time periods $\mathcal{T} = \{1, 2\}$. Locations 1 and 2 have a reward of 100 (*i.e.*, $r_1 = r_2 = 100$), whereas location 3 has a reward of 51 (*i.e.*, $r_3 = 51$). Customer A is willing to attend locations 1 and 3 (*i.e.*, $a_{1A} = a_{3A} = 1, a_{2A} = 0$), whereas customer B is willing to attend locations 2 and 3 (*i.e.*, $a_{2B} = a_{3B} = 1, a_{1B} = 0$). Both customers have 1 unit of spawning demand at each time period (*i.e.*, $d_j^t = 1 \forall j \in \{A, B\}, \forall t \in \{1, 2\}$).

The optimal (integer) solution is to install the temporary facility at location 1 at time period $t = 1$ and location 2 at time period $t = 2$, which gives an optimal (integer) objective value $\Pi^* = 300$. The \overline{DI} Formulation has an optimal objective value of $\Pi^1 = 300$ (*i.e.*, the tightest bound possible), whereas the \overline{SI} Formulation has an optimal objective value of $\Pi^M = 302$ (*i.e.*, a looser bound). Instead of opening location 2 at time period $t = 2$ completely (*i.e.*, setting $y_2^2 = 1$), the \overline{SI} Formulation opens (*i*) half of location 2 ($y_2^2 = \frac{1}{2}$) to capture some demand from customer B with a higher reward, and (*ii*) half of location 3 ($y_3^2 = \frac{1}{2}$) to capture remaining demand from customers A and B . Therefore, the DI Formulation is strictly tighter than the SI Formulation, and provides a tighter continuous relaxation than the SI Formulation. \square

B.2 Proof of Theorem 2

We first formalize the decision version of the SPP, and then present the proof.

Decision version of the SPP:

INSTANCE: A finite collection of n sets $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$, a finite set of m elements $\mathcal{B} = \{B_1, \dots, B_m\}$ appearing in \mathcal{C} and a positive integer $1 \leq K \leq |\mathcal{C}|$.
 QUESTION: Is there at least K mutually disjoint sets in \mathcal{C} ?

Proof. First, we show that the CDSFLP-CCD is in NP. A decision problem is in NP if the certificate answering the decision question can be verified in polynomial time. For the CDSFLP-CCD, this means verifying if $\pi(\mathbf{y}) \geq \Pi$ for some location sequence \mathbf{y} . Proposition 1 guarantees that $\pi(\mathbf{y})$ can be computed in polynomial time, and it suffices to check whether $\pi(\mathbf{y}) \geq \Pi$ to answer the decision question. Therefore, it follows that the CDSFLP-CCD is in NP.

Second, we show that the CDSFLP-CCD is NP-hard by reducing the SPP to it. By showing that if there is a certificate satisfying the SPP decision question, then there is a certificate satisfying the CDSFLP-CCD decision question (referred to as forward direction) and vice-versa (referred to as backward direction), it holds that the CDSFLP-CCD is NP-hard.

Reduction. Consider the CDSFLP-CCD instance built from an SPP instance with $2n$ locations, $m + n$ customers, and n time periods as follows. Each element B_j generates an element-customer j with spawning demands $d_j^1 = 1, d_j^t = 0, \forall t \in \mathcal{T} : t \geq 2$. Each set \mathcal{C}_i generates an authentic location i_1 with a reward of $r_{i_1} = \frac{1-\delta}{|\mathcal{C}_i|}$, where $0 < \delta < 1$; a fictive location i_2 with a reward of $r_{i_2} = \epsilon$, where $\epsilon = \frac{M+1}{2}$ and $M = \max_{\mathcal{C}_i \in \mathcal{C}} \left\{ \frac{|\mathcal{C}_i|-1}{|\mathcal{C}_i|} (1-\delta) \right\}$; and a set-customer j with spawning demands $d_j^1 = 1, d_j^t = 0, \forall t \in \mathcal{T} : t \geq 2$. We build preference rules a_{ij} so that each authentic location i_1 captures all element-customers that belong to set \mathcal{C}_i , and each fictive location i_2 captures solely the respective set-customer.

We remark that the reward for a time period t is always between ϵ (*i.e.*, if the provider chooses a fictive location that captures the demand unit from the respective set-customer) and $(1-\delta)$ (*i.e.*, if the provider chooses an authentic location that captures the demand unit from all element-customers in the respective set). Note that, in a given time period t , the provider prefers a fictive location over an authentic one if the latter cannot capture the demand unit from all element-customers in the respective set. As a direct implication, if K is the maximum number of disjoint sets in \mathcal{C} , the total reward over the planning horizon is at most $K(1-\delta) + (n-K)\epsilon$, where $K(1-\delta)$ units come from authentic locations describing mutually disjoint sets and $(n-K)\epsilon$ units come from fictive locations. We look therefore for an objective value Π of the CDSFLP-CCD instance equal to $K(1-\delta) + (n-K)\epsilon$.

Forward direction. Assume that there is a certificate $\mathcal{D} = \{\mathcal{C}_{h(1)}, \dots, \mathcal{C}_{h(K)}\}$ to the SPP decision question, *i.e.*, a subcollection \mathcal{D} with at least K mutually disjoint sets, where function $h(t)$ maps the t -th set in the certificate to its index in collection \mathcal{C} . We show that this certificate implies the existence of a certificate to the CDSFLP-CCD decision question, *i.e.*, a location sequence \mathbf{y} for the CDSFLP-CCD instance with a total reward of at least $\Pi = K(1-\delta) + (n-K)\epsilon$.

In the associated solution of the CDSFLP-CCD, the provider opens authentic locations linked to sets $\mathcal{C}_{h(t)}$ for time periods $t \in \{1, \dots, K\}$, and fictive locations for the remaining time periods $t \in \{K+1, \dots, n\}$. Since sets $\mathcal{C}_{h(1)}, \dots, \mathcal{C}_{h(K)}$ are mutually disjoint, we obtain a reward of $(1-\delta)$ from each time period between 1 and K , and a reward of ϵ from each time period between $K+1$ and n , which yields a total reward of $K(1-\delta) + (n-K)\epsilon$. Thus, the existence of a certificate \mathcal{D} to the SPP decision question implies the existence of a certificate \mathbf{y} for the CDSFLP-CCD decision question.

Backward direction. Assume that there is no certificate $\mathcal{D} = \{\mathcal{C}_{h(1)}, \dots, \mathcal{C}_{h(K)}\}$ to the SPP decision question, *i.e.*, no subcollection \mathcal{D} with at least K mutually disjoint sets. We show that this implies no location sequence \mathbf{y} for the DSFLP-CCD instance with a total reward of at least $\Pi = K(1 - \delta) + (n - K)\epsilon$.

We know from the forward direction that, if the SPP instance has at least K mutually disjoint sets, then there is a location sequence \mathbf{y} such that the total reward of the associated CDSFLP-CCD instance is $\Pi = K(1 - \delta) + (n - K)\epsilon$. Moreover, from the implication at the start of this proof, if the maximum number of mutually disjoint sets for the SPP instance is at most $K' < K$, then the optimal value of the associated CDSFLP-CCD instance is $\Pi = K'(1 - \delta) + (n - K')\epsilon < K(1 - \delta) + (n - K)\epsilon$. Hence, if there are no K mutually disjoint sets in subcollection \mathcal{D} , then there cannot be a location sequence \mathbf{y} for the associated CDSFLP-CCD instance with a total reward of at least $\Pi = K(1 - \delta) + (n - K)\epsilon$.

Since both directions hold, the CDSFLP-CCD is in fact NP-hard. We now show that the CDSFLP-CCD is as inapproximable as the SPP through the so-called gap technique (Schuurman and Woeginger, 2001). Let $obj_{dSPP}(\mathcal{C}^{ANSWER})$ be the optimal value assigned to an instance \mathcal{C}^{ANSWER} of the decision version of the SPP, and $obj_{CCD}(f(\mathcal{C}^{ANSWER}))$ be the optimal value assigned to the associated CDSFLP-CCD instance $f(\mathcal{C}^{ANSWER})$, which can be determined in polynomial time through the reduction presented earlier. Let \mathcal{C}^{YES} and \mathcal{C}^{NO} denote instances of the decision version of the SPP with the same \mathcal{C} but with different target outcomes: the former resulting in *YES* and the latter resulting in *NO*. Since the SPP cannot be approximated within a factor $|\mathcal{C}|^{1-\alpha}$ for any $\alpha > 0$ (Ausiello et al., 1980; Hastad, 1996; Hazan et al., 2006), it holds that $\frac{obj_{dSPP}(\mathcal{C}^{NO})}{obj_{dSPP}(\mathcal{C}^{YES})} < |\mathcal{C}|^{1-\alpha}$ for any $\alpha > 0$. We now analyze the ratio $\frac{obj_{CCD}(f(\mathcal{C}^{NO}))}{obj_{CCD}(f(\mathcal{C}^{YES}))}$ for the CDSFLP-CCD to determine if we can answer the decision version of the SPP through the CDSFLP-CCD, where $obj_{CCD}(f(\mathcal{C})) = obj_{dSPP}(\mathcal{C})(1 - \delta - \epsilon) + n\epsilon$ holds by construction:

$$\frac{obj_{CCD}(f(\mathcal{C}^{NO}))}{obj_{CCD}(f(\mathcal{C}^{YES}))} = \frac{obj_{dSPP}(\mathcal{C}^{NO})(1 - \delta - \epsilon) + n\epsilon}{obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon) + n\epsilon} < \frac{|\mathcal{C}|^{1-\alpha} obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon) + n\epsilon}{obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon) + n\epsilon}.$$

We can then simplify the right-hand side by employing the fact that, if $a > b$, then $g(x) = \frac{a+x}{b+x} < \frac{a}{b}$:

$$\frac{|\mathcal{C}|^{1-\alpha} obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon) + n\epsilon}{obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon) + n\epsilon} < \frac{|\mathcal{C}|^{1-\alpha} obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon)}{obj_{dSPP}(\mathcal{C}^{YES})(1 - \delta - \epsilon)} < |\mathcal{C}|^{1-\alpha} = T^{1-\alpha}$$

Therefore, the CDSFLP-CCD also cannot be approximated within a factor $T^{1-\alpha}$, unless P = NP. \square

B.3 Proof of Theorem 3

We define the 3SAT, and then present the proof.

3SAT:

INSTANCE: A finite set of n Boolean variables $\mathcal{B} = \{B_1, \dots, B_n\}$ and a finite set of m clauses $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_m\}$, each with exactly three variables.

QUESTION: Is there a literal assignment such that all clauses are satisfied?

Proof. The CDSFLP-CCD with identical rewards is in NP as the general CDSFLP-CCD is in NP (see Theorem 2). We show here that the CDSFLP-CCD with identical rewards is NP-hard by reducing the 3SAT to it. By showing that if there is a certificate satisfying the 3SAT, then there is a certificate satisfying the CDSFLP-CCD decision question (referred to as forward direction) and vice-versa (referred to as backward direction), it holds that the CDSFLP-CCD with identical rewards is NP-hard.

Reduction. By hypothesis, $r_i = R, \forall i \in \mathcal{I}$ and $R \in \mathbb{Q}^+$. Now, consider the CDSFLP-CCD instance with identical rewards built from a 3SAT instance with $2n$ locations, one for each assignment of a literal to a variable (*e.g.*, $[x_1, true]$ and $[x_1, false]$ are two different locations); $n+m$ customers, one for each variable and each clause; and n time periods. We set $d_j^1 = 1, d_j^t = 0, \forall t \in \mathcal{T} : t \geq 2$ and build preference rules a_{ij} so that each location (*i.e.*, assignment of literal to variable) captures satisfied clause-customers (*i.e.*, customers originated from clauses) and the respective variable-customer (*i.e.*, customers originated from the variables). Note that the total reward for the planning horizon is at most $R(n+m)$, which can only be achieved in n time periods if each variable-customer is captured at least once (*i.e.*, each variable has been assigned to a literal), as well as each clause-customer is captured once (*i.e.*, each clause has been satisfied through some assignment). In fact, we look for an objective value Π of the CDSFLP-CCD with identical rewards equal to $R(n+m)$, which guarantees that there is a literal assignment that satisfies all clauses.

Forward direction. Assume that there is a certificate to the 3SAT, *i.e.*, an assignment of literals to variables in \mathcal{B} such that the clauses in \mathcal{C} are satisfied. We show that this certificate implies the existence of a certificate to the CDSFLP-CCD decision question, *i.e.*, a location sequence \mathbf{y} for the CDSFLP-CCD instance with a total reward of at least $\Pi = R(n+m)$.

In the associated solution of the CDSFLP-CCD, the provider opens, at time period t , the location linked to the t -th variable and its literal. Since the assignment satisfies all clauses, each clause-customer and each variable-customer is captured at least once over the planning horizon, yielding a total reward of $R(n+m)$. Thus, the existence of a certificate to the 3SAT implies the existence of a certificate \mathbf{y} for the CDSFLP-CCD decision question.

Backward direction. Assume that there is a certificate \mathbf{y} to the CDSFLP-CCD decision question, *i.e.*, a location sequence \mathbf{y} with a total reward of at least $\Pi = R(n+m)$. We show that this certificate implies the existence of a certificate to the 3SAT, *i.e.*, an assignment of literals to variables in \mathcal{B} such that the clauses in \mathcal{C} are satisfied.

The backward assumption guarantees that the total reward is at least $\Pi = R(n+m)$. In turn, the only way to obtain such a solution is by capturing each clause-customer and each variable-customer at least once throughout the planning horizon, which means that each variable is assigned to a literal while satisfying all clauses. Thus, the existence of a certificate \mathbf{y} to the CDSFLP-CCD decision question implies the existence of a certificate of the 3SAT.

Since both directions hold, the CDSFLP-CCD with identical rewards is in fact NP-hard. In addition, since the 3SAT is strongly NP-hard (Garey and Johnson, 1979), the same result holds for the CDSFLP-CCD. \square

B.4 Proof of Theorem 4

For the sake of simplicity, we abuse the notation by writing $\mathcal{J}(i_1^B, \dots, i_T^B)$ instead of $\mathcal{J}(\{i_1^B, \dots, i_T^B\})$ for the set of customers captured by locations i_1^B, \dots, i_T^B . Based on Algorithm 1, we define the

location i_t^B chosen by the Backward Greedy Heuristic at time period t as the one with the highest marginal contribution:

$$i_t^B = \arg \max_{i \in \mathcal{I}} \left\{ \sum_{\substack{s \in \mathcal{T}: \\ s \leq t}} \sum_{\substack{j \in \mathcal{J}(i): \\ j \notin \mathcal{J}(i_{t+1}^B, \dots, i_T^B)}} d_j^s \right\} \quad (12)$$

and the total reward of the heuristic location sequence i_1^B, \dots, i_T^B as follows:

$$\pi(\mathbf{y}^B) = R \left[\sum_{j \in \mathcal{J}(i_T^B)} \sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} d_j^s + \sum_{\substack{j \in \mathcal{J}(i_{T-1}^B): \\ j \notin \mathcal{J}(i_T^B)}} \sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} d_j^s + \dots + \sum_{\substack{j \in \mathcal{J}(i_1^B): \\ j \notin \mathcal{J}(i_2^B, \dots, i_T^B)}} \sum_{\substack{s \in \mathcal{T}: \\ s \leq 1}} d_j^s \right]$$

Proof. We analyze the difference $\pi(\mathbf{y}^*) - \pi(\mathbf{y}^B)$ to upper bound it by $\pi(\mathbf{y}^B)$. We rewrite first the total reward as a function of spawning demands d_j^s , grouping them by time period s , and the identical reward R :

$$\begin{aligned} \pi(\mathbf{y}^*) - \pi(\mathbf{y}^B) &= R \left[\sum_{j \in \mathcal{J}(i_1^*, i_2^*, \dots, i_T^*)} d_j^1 + \sum_{j \in \mathcal{J}(i_2^*, \dots, i_T^*)} d_j^2 + \dots + \sum_{j \in \mathcal{J}(i_{T-1}^*, i_T^*)} d_j^{T-1} + \sum_{j \in \mathcal{J}(i_T^*)} d_j^T \right] - \\ &R \left[\sum_{j \in \mathcal{J}(i_1^B, i_2^B, \dots, i_T^B)} d_j^1 + \sum_{j \in \mathcal{J}(i_2^B, \dots, i_T^B)} d_j^2 + \dots + \sum_{j \in \mathcal{J}(i_{T-1}^B, i_T^B)} d_j^{T-1} + \sum_{j \in \mathcal{J}(i_T^B)} d_j^T \right] = (\#^1). \end{aligned}$$

We can further upper bound this difference by taking spawning demands that appear with a positive coefficient after executing the subtraction by time periods as follows:

$$(\#^1) \leq R \left[\sum_{\substack{j \in \mathcal{J}(i_1^*, i_2^*, \dots, i_T^*): \\ j \notin \mathcal{J}(i_1^B, i_2^B, \dots, i_T^B)}} d_j^1 + \sum_{\substack{j \in \mathcal{J}(i_2^*, \dots, i_T^*): \\ j \notin \mathcal{J}(i_2^B, \dots, i_T^B)}} d_j^2 + \dots + \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*, i_T^*): \\ j \notin \mathcal{J}(i_{T-1}^B, i_T^B)}} d_j^{T-1} + \sum_{\substack{j \in \mathcal{J}(i_T^*): \\ j \notin \mathcal{J}(i_T^B)}} d_j^T \right] = (\#^2).$$

We now isolate spawning demands related to location i_t^* for each time period t , and show that they are upper bounded by the contribution of location i_t^B to the heuristic location sequence i_1^B, \dots, i_T^B . We start with time period T , where $\sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{\substack{j \in \mathcal{J}(i_T^*): \\ j \notin \mathcal{J}(i_T^B)}} d_j^s \leq \sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{j \in \mathcal{J}(i_T^*)} d_j^s$ holds trivially and $\sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{j \in \mathcal{J}(i_T^*)} d_j^s \leq \sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{j \in \mathcal{J}(i_T^B)} d_j^s$ holds thanks to Equation (12):

$$\begin{aligned} (\#^2) &= R \left[\sum_{\substack{j \in \mathcal{J}(i_1^*, i_2^*, \dots, i_{T-1}^*): \\ j \notin \mathcal{J}(i_1^B, i_2^B, \dots, i_T^B)}} d_j^1 + \sum_{\substack{j \in \mathcal{J}(i_2^*, \dots, i_{T-1}^*): \\ j \notin \mathcal{J}(i_2^B, \dots, i_T^B)}} d_j^2 + \dots + \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*): \\ j \notin \mathcal{J}(i_{T-1}^B, i_T^B)}} d_j^{T-1} + \sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{\substack{j \in \mathcal{J}(i_T^*): \\ j \notin \mathcal{J}(i_T^B)}} d_j^s \right] \leq \\ &R \left[\sum_{\substack{j \in \mathcal{J}(i_1^*, i_2^*, \dots, i_{T-1}^*): \\ j \notin \mathcal{J}(i_1^B, i_2^B, \dots, i_T^B)}} d_j^1 + \sum_{\substack{j \in \mathcal{J}(i_2^*, \dots, i_{T-1}^*): \\ j \notin \mathcal{J}(i_2^B, \dots, i_T^B)}} d_j^2 + \dots + \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*): \\ j \notin \mathcal{J}(i_{T-1}^B, i_T^B)}} d_j^{T-1} \right] + R \left[\sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{j \in \mathcal{J}(i_T^B)} d_j^s \right] = (\#^3). \end{aligned}$$

We do the same for time period $T-1$, where $\sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*): \\ j \notin \mathcal{J}(i_{T-1}^B, i_T^B)}} d_j^s \leq \sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*): \\ j \notin \mathcal{J}(i_T^B)}} d_j^s$ holds trivially and $\sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*): \\ j \notin \mathcal{J}(i_T^B)}} d_j^s \leq \sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} \sum_{\substack{j \in \mathcal{J}(i_{T-1}^B): \\ j \notin \mathcal{J}(i_T^B)}} d_j^s$ holds thanks to Equation (12):

$$\begin{aligned}
 (\#^3) = R & \left[\sum_{\substack{j \in \mathcal{J}(i_1^*, i_2^*, \dots, i_{T-2}^*): \\ j \notin \mathcal{J}(i_1^B, i_2^B, \dots, i_T^B)}} d_j^1 + \sum_{\substack{j \in \mathcal{J}(i_2^*, \dots, i_{T-2}^*): \\ j \notin \mathcal{J}(i_2^B, \dots, i_T^B)}} d_j^2 + \dots + \sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} \sum_{\substack{j \in \mathcal{J}(i_{T-1}^*): \\ j \notin \mathcal{J}(i_{T-1}^B, i_T^B)}} d_j^s \right] + R \left[\sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{\substack{j \in \mathcal{J}(i_T^B)}} d_j^s \right] \leq \\
 & R \left[\sum_{\substack{j \in \mathcal{J}(i_1^*, i_2^*, \dots, i_{T-2}^*): \\ j \notin \mathcal{J}(i_1^B, i_2^B, \dots, i_T^B)}} d_j^1 + \sum_{\substack{j \in \mathcal{J}(i_2^*, \dots, i_{T-2}^*): \\ j \notin \mathcal{J}(i_2^B, \dots, i_T^B)}} d_j^2 + \dots \right] + R \left[\sum_{\substack{s \in \mathcal{T}: \\ s \leq T-1}} \sum_{\substack{j \in \mathcal{J}(i_{T-1}^B): \\ j \notin \mathcal{J}(i_T^B)}} d_j^s + \sum_{\substack{s \in \mathcal{T}: \\ s \leq T}} \sum_{\substack{j \in \mathcal{J}(i_T^B)}} d_j^s \right].
 \end{aligned}$$

It is easy to see that repeating this reasoning gradually builds the total reward $\pi(\mathbf{y}^B)$ of the heuristic location sequence i_1^B, \dots, i_T^B on the right-hand side. Therefore, it holds that $\pi(\mathbf{y}^*) - \pi(\mathbf{y}^B) \leq \pi(\mathbf{y}^B)$, and the backward greedy heuristic is a 2-approximation algorithm for the CDSFLP-CCD with identical rewards. \square

B.5 Proof of Theorem 5

Lemma 1 *Let i_1, \dots, i_T be a feasible solution of a CDSFLP-CCD instance with loyal customers. If there is a location $i' \in \mathcal{I}$ chosen for two or more time periods of the planning horizon (i.e., $\exists \mathcal{T}' = \{t'_1, t'_2, \dots, t'_{K-1}, t'_K\} \subseteq \mathcal{T}$ such that $i_t = i' \forall t \in \mathcal{T}'$), there is an equivalent solution (i.e., with the same total reward) $\tilde{i}_1, \dots, \tilde{i}_T$ without repetition where $\tilde{i}_t = \emptyset \forall t \in \mathcal{T}'$, $\tilde{i}_t = i_t \forall t \in \mathcal{T} \setminus \mathcal{T}'$ and $\tilde{i}_{t'_K} = i'$.*

Proof. Let i' be the repeated location. Under loyal customers, we can split the total reward into contributions from location i' and those from other locations, denoted by Π , because there is no intersection between customers captured by location i' and those captured by other locations. This reasoning gives

$$\begin{aligned}
 \pi(i_1, \dots, i_T) &= \Pi + \sum_{t \in \mathcal{T}'} \rho(i', t \mid i_1, \dots, i_T) = \\
 & \Pi + r_{i'} \sum_{j \in \mathcal{J}(i')} \left(\sum_{\substack{s \in \mathcal{T}: \\ s > 0 \\ s \leq t'_1}} d_j^s + \sum_{\substack{s \in \mathcal{T}: \\ s > t'_1 \\ s < t'_2}} d_j^s + \dots + \sum_{\substack{s \in \mathcal{T}: \\ s > t'_{K-1} \\ s \leq t'_K}} d_j^s \right) = \\
 & \Pi + r_{i'} \sum_{j \in \mathcal{J}(i')} \sum_{\substack{s \in \mathcal{T}: \\ s \leq t'_K}} d_j^s = \pi(\tilde{i}_1, \dots, \tilde{i}_T). \square
 \end{aligned}$$

Corollary 1 *Let i_1, \dots, i_T be a feasible solution of a CDSFLP-CCD instance with loyal customers. We assume that this feasible solution has no repetition without loss of generality. The marginal reward function can be rewritten as $\rho(i_t, t) = r_{i_t} \sum_{j \in \mathcal{J}(i_t)} \sum_{\substack{s \in \mathcal{T}: \\ s \leq t}} d_j^s$, and the total reward function can be rewritten as $\pi(i_1, \dots, i_T) = \sum_{t \in \mathcal{T}} \rho(i_t, t)$.*

Proof. This is a direct outcome of Lemma 1. \square

We are now ready to present the proof of Theorem 5.

Proof. Each feasible solution i_1, \dots, i_T of the CDSFLP-CCD with loyal customers has an equivalent solution without repetition, which is nothing but an exact assignment of locations to time periods. Thus, we can solve this CDSFLP-CCD instance by solving an assignment problem in polynomial time (Kuhn, 1955). Let $E = I - T$ be the difference between the number of candidate locations and time periods. We can create an instance of the assignment in polynomial time by setting the weight of assigning candidate location i to time period t as $\rho(i, t)$. If $E = 0$, we do not have to conduct further adaptations. If $E < 0$, we need to add $|E|$ virtual candidate locations, such that their weight is 0 to all time periods. Similarly, if $E > 0$, we need to add $|E|$ virtual time periods, such that their weight is 0 to all candidate locations. The feasible solution can be naturally drawn from the assignment of (true) candidate locations to (true) time periods in polynomial time. Thus, the CDSFLP-CCD with loyal customers is polynomially solvable. \square

B.6 Proof of Theorem 6

We first rewrite the dual subproblem $w_j^D(\mathbf{y})$ as follows, which modifies only Constraints (6b):

$$\min_{\mathbf{p}, \mathbf{q}} \quad \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}} a_{ij} y_i^t p_i^t + q^0 \quad (13a)$$

$$\text{s.t.} \quad p_i^t \geq \max_{\ell \in \mathcal{T}^S: \ell < t} \{r_i D_j^{\ell t} - q^\ell + q^t\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T} \quad (13b)$$

$$q^\ell \geq 0 \quad \forall \ell \in \mathcal{T}^S \quad (13c)$$

$$p_i^t \in \mathbb{R} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T} \quad (13d)$$

$$q^t \in \mathbb{R} \quad \forall t \in \mathcal{T}^S. \quad (13e)$$

Lemma 2 *In an optimal solution of the dual subproblem $w_j^D(\mathbf{y})$, Constraints (13b) may be posed with equality without changing the optimal objective value.*

Proof. Let \mathbf{x}^* be the optimal solution of the primal subproblem $w_j^P(\mathbf{y})$, which can be computed in polynomial time based on Proposition 1. If $y_i^t = 0$, variable p_i^t does not appear in Objective Function (13a), and we can satisfy Constraints (13b) for i and t with equality without changing the objective value for the term i and t of the sum. If $y_i^t = 1$, there are two cases. On the one hand, if $a_{ij} = 0$, variable p_i^t also does not appear in Objective Function (13a), and the previous reasoning applies. On the other hand, if $a_{ij} = 1$, we know that $x_i^{st^*} = 1$ for some time period $s < t$ through Constraints (5c). Then, by complementary slackness, Constraints (13b) must be satisfied for i and t with equality (i.e., if $x_i^{st^*} = 1$, then $p_i^t = r_i D_j^{st} - q^s + q^t$). Therefore, Constraints (13b) may be posed with equality without changing the optimal objective value. \square

Lemma 2 allow us to project out variables p_i^t out of the dual subproblem $w_j^D(\mathbf{y})$, and obtain an equivalent form for Constraints (13b). In other words, from the proof of Lemma 2, we have:

$$\begin{aligned} p_i^t &\geq \max_{\ell \in \mathcal{T}^S: \ell < t} \{r_i D_j^{\ell t} - q^\ell + q^t\} && \forall i \in \mathcal{I}, \forall t \in \mathcal{T} \\ \implies r_i D_j^{st} - q^s + q^t &\geq \max_{\ell \in \mathcal{T}^S: \ell < t, \ell \neq s} \{r_i D_j^{\ell t} - q^\ell + q^t\} && \forall i \in \mathcal{I}, \forall t \in \mathcal{T}, \forall s \in \mathcal{T}^S: \\ &&& s < t, x_i^{st^*} = 1 \\ \implies r_i D_j^{st} - q^s + q^t &\geq r_i D_j^{\ell t} - q^\ell + q^t && \forall i \in \mathcal{I}, \forall t \in \mathcal{T}, \forall s \in \mathcal{T}^S, \forall \ell \in \mathcal{T}^S: \\ &&& s < t, x_i^{st^*} = 1, \ell < t, \ell \neq s \end{aligned}$$

which gives the following reduced but equivalent dual subproblem $w_j^R(\mathbf{y})$:

$$\min_{\mathbf{q}} \quad q^0 \quad (14a)$$

$$\text{s.t.:} \quad q^\ell \geq r_i(D_j^{\ell t} - D_j^{st}) + q^s \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}, \forall s \in \mathcal{T}^S, \forall \ell \in \mathcal{T}^S: \quad (14b)$$

$$s < t, x_i^{st^*} = 1, \ell < t, \ell \neq s$$

$$q^\ell \geq 0 \quad \forall \ell \in \mathcal{T}^S \quad (14c)$$

$$q^\ell \in \mathbb{R} \quad \forall \ell \in \mathcal{T}^S \quad (14d)$$

from which variables p_i^t can be computed based on variables q^t as follows:

$$p_i^t = \max_{\ell \in \mathcal{T}^S: \ell < t} \{r_i D_j^{\ell t} - q^\ell + q^t\} \quad \forall i \in \mathcal{I}, \forall t \in \mathcal{T}. \quad (15)$$

We can now easily derive an analytical formula to compute a feasible solution for this reduced dual subproblem $w^R(\mathbf{y})$ based on Constraints (14b)–(14c):

$$q^\ell = \max_{\substack{i \in \mathcal{I}, s \in \mathcal{T}^S, t \in \mathcal{T}: \\ s < t, x_i^{st^*} = 1 \\ \ell < t, \ell \neq s}} \{r_i(D_j^{\ell t} - D_j^{st}) + q^s, 0\} \quad \forall \ell \in \mathcal{T}^S. \quad (16)$$

The main challenge here is that Equation (16) requires variable q^s to compute variable q^ℓ . In this sense, we must ensure that there is a feasible order to compute them independently (*i.e.*, without some sort of recurrent definition). Let $\mathcal{T}^+(\mathbf{y}) = \{\ell \in \mathcal{T} \mid \sum_{i \in \mathcal{I}} a_{ij} y_i^\ell = 1\} \cup \{0\}$ and $\mathcal{T}^-(\mathbf{y}) = \mathcal{T} \setminus \mathcal{T}^+(\mathbf{y})$ be the set of time periods where customer j has been captured (+) or remained free (−), respectively, in location sequence \mathbf{y} .

Lemma 3 Equation (16) provides feasible values for variables q^ℓ in the dual subproblem $w^R(\mathbf{y})$ if we first compute them for time periods $\ell \in \mathcal{T}^+(\mathbf{y})$ in decreasing order, then for time periods $\ell \in \mathcal{T}^-(\mathbf{y})$.

Proof. We show first, by contradiction, that there is no recurrent definition in the first pass, where we compute variable q^ℓ for time periods $\ell \in \mathcal{T}^+(\mathbf{y})$ in decreasing order. Assume, for the sake of contradiction, that Equation (16) needs the value of some variable q^s to compute some variable $\ell \in \mathcal{T}^+(\mathbf{y})$ that has not been computed yet. This implies that (i) $x_{i_1}^{\ell t^*} = 1$ for some time period t and some location i_1 since $\ell \in \mathcal{T}^+(\mathbf{y})$, and (ii) $x_{i_2}^{st^*} = 1$ for some location i_2 since q^s appears in Equation (16), which is absurd. In other words, we cannot have $x_{i_1}^{\ell t^*} = 1$ and $x_{i_2}^{st^*} = 1$ without violating flow conservation of the primal subproblem $w^P(\mathbf{y})$. Therefore, the lemma is correct for $\ell \in \mathcal{T}^+(\mathbf{y})$ (*i.e.*, we can determine feasible values for q^ℓ for time periods $\ell \in \mathcal{T}^+(\mathbf{y})$ by applying Equation (16) in decreasing order). We consider now the second pass, where we compute variable q^ℓ for time periods $\ell \in \mathcal{T}^-(\mathbf{y})$. Note that we only employ variables q^s with $s \in \mathcal{T}^+(\mathbf{y})$ in Equation (16), which were already computed in the first pass. Therefore, the lemma holds. \square

Lemma 4 Let (\mathbf{p}, \mathbf{q}) be a feasible solution of the dual subproblem $w_j^D(\mathbf{y})$ computed with Equations (16) and (15). This solution is, in fact, optimal.

Proof. We show the optimality of this feasible solution by strong duality (*i.e.*, by showing that the objective value of the dual solution (\mathbf{p}, \mathbf{q}) is equal to the objective value $\sum_{t \in \mathcal{T}} \sum_{\substack{\ell \in \mathcal{T}: \\ \ell < t}} r_i D_j^{\ell t} x_i^{\ell t^*}$ of the optimal primal solution \mathbf{x}^*). Lemma 2 shows that only variables p_i^t such that $y_i^t = 1$ and $a_{ij} = 1$ appear in Objective Function (13a). In turn, each variable p_i^t that appears in the objective

function is equal to $r_i D_j^{st} - q^s + q^t$ for some $x_i^{st*} = 1$ due to complementary slackness. Plugging this information into Objective Function (13a) gives $\sum_{t \in \mathcal{T}} \sum_{\substack{\ell \in \mathcal{T}: \\ \ell < t}} r_i D_j^{\ell t} x_i^{\ell t*} + q^{L(\mathbf{y})}$, where $L(\mathbf{y})$ is the last time period where customer j was captured in location sequence \mathbf{y} . More precisely, variables q^s and q^t in the definition of each variable p_i^t cancel each other through the sum, with the exception of $q^{L(\mathbf{y})}$. Note that, by construction, Equations (16) always sets $q^{L(\mathbf{y})} = 0$. Therefore, by strong duality, this feasible solution (\mathbf{p}, \mathbf{q}) is, in fact, optimal. \square

Theorem 6 naturally holds as an outcome of Lemmas 2, 3, and 4.

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