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Robust Facility Location Under Demand Uncertainty and Facility Disruptions

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Abstract. Facility location decision is strategic: the construction of a new facility is typically costly and the impact of the decision is long-lasting. Environmental changes, such as population shift and natural disasters, may cause today's optimal decision to perform poorly in the future. Thus, it is important to consider potential uncertainties in the design phase, while explicitly taking into account the possible customer reassignments as recourse decisions in the execution phase. This paper studies a robust fixed-charge location problem under uncertain demand and facility disruptions. To model this problem, we adopt a two-stage robust optimization framework, where the first-stage location decision is made `here-and-now' and the second-stage allocation decision can be deferred until the uncertainty information is revealed. We develop a column-and-constraint generation algorithm to solve the models exactly, and use the affine policy (AP) to generate approximate solutions. We further develop two solution methods for the models based on the AP. We conduct extensive numerical tests to study the impact of uncertainties on solution configuration and algorithm efficiency. The performance of the robust models is also measured against that of the two-stage stochastic programming model.

Keywords. Location, uncertain demand, disruption risk, robust optimization.

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1 Introduction

Facility location is an important aspect of strategic planning for both private companies and public sectors. Whether a manufacturer building a new plant, or a city planner choosing locations for public facilities, planners are often challenged by difficult resource allocation decisions (Owen and Daskin 1998). The construction and acquisition of a new facility is typically a costly and time-consuming process. Therefore, once a new facility is built, it is expected to remain in operation for several years. However, environmental changes, such as population shift and transportation infrastructure issues, may turn today's optimal decision into tomorrow's poor performance. It is therefore critical to consider potential uncertainties at the planning stage, to avoid high recourse costs at the operational stage.

In facility location problems, the uncertainty can be generally classified into three types: provider-side uncertainty, receiver-side uncertainty, and in-between uncertainty (Shen et al. 2011). The provider-side uncertainty includes uncertain facility capacity and status (operational or failed). The receiver-side uncertainty captures randomness in demand. The in-between uncertainty involves uncertain transportation costs/time and arc status. These three types of uncertainty have been widely considered in the literature. For example, facility location under demand uncertainty (Atamtürk and Zhang 2007; Baron et al. 2011; Gülpınar et al. 2013), facility location under random travel costs/times (Nikoofal and Sadjadi 2010; Gao and Qin 2016; Mišković et al. 2017), and facility location under disruption risks (Cui et al. 2010; Yu et al. 2017; Afify et al. 2019).

Most studies consider one type of uncertainty at a time whereas a few of them consider simultaneous uncertainties, which are common in real-world applications of facility location where facilities are faced with multiple uncertainties at the same time. For example, at the time of building a new facility, it is typically very challenging to precisely estimate the future demand of multiple customers over long-term horizons. In addition, during the operational stage, a facility could be disrupted by various risks such as power outages and natural disasters. Thus, in this work, we study a capacitated fixed-charge location problem (CFLP) that considers supply-side and receiver-side uncertainties simultaneously. More specifically, customers are subject to demand uncertainty and facilities may experience disruption risks. We adopt a two-stage robust optimization (RO) scheme to model the problem, which does not rely on any probability information, because it is typically difficult to estimate the probability in this strategic context. To solve the robust models, we develop both exact and heuristic methods.

Our contributions. We consider this study to make the following contributions: (1) To the best of our knowledge, this work is the first to study the CFLP with simultaneous provider-side and receiver-side uncertainties in the two-stage RO framework. The corresponding model generalizes the problems with only demand uncertainty and with only facility disruptions. (2) We develop both exact (column-and-constraint generation, C&CG) and heuristic (affine policy, AP) methods to solve the robust models. For the model based on the AP, we further use a row generation (RG) algorithm besides directly solving the dualized reformulation. We identify conditions under which the AP generates optimal first-stage solutions for the robust models. (3) We conduct extensive numerical tests to study the differences in solutions produced by the three robust models, the impact of uncertainty, and the efficiency of algorithms. We also benchmark the two-stage RO framework against the two-stage stochastic programming method.

The rest of this paper is organized as follows. Section 2 presents related literature. Section 3 constructs the deterministic and robust models. Section 4 describes both exact and heuristic solution methods. Section 5 discusses the numerical results, and Section 6 concludes the paper.

2 Related literature

This section reviews related work. A summary of the papers is given in Table 1.

For early work of facility location under demand uncertainty, see the review paper by Snyder (2006). Baron et al. (2011) study a multi-period facility location problem under demand uncertainty, where a box uncertainty set and an ellipsoid uncertainty set are used. Atamtürk and Zhang (2007) are the first to study the two-stage robust location-transportation problem (LTP), and a cutting plane algorithm is applied. They compare solutions generated by the two-stage RO method, one-stage (also known as static) RO method, and the stochastic optimization method. For the same problem, Zeng and Zhao (2013) focus on comparing the performance of the C&CG algorithm and the Benders-style cutting plane method. Ardestani-Jaafari and Delage (2017) study a multi-period LTP and develop various approximation schemes to solve the problem based on the AP.

Gao and Qin (2016) study a p-hub center location problem under uncertain travel times. A chance constrained programming (CCP) approach is used and the deterministic equivalent model is solved by a genetic algorithm. Mišković et al. (2017) consider a two-echelon facility location problem, where products are first delivered to depots and then from depots to customers. The transportation costs in both echelons are uncertain. They use budgeted uncertainty sets and a static RO method for the problem. Matthews et al. (2019) address a single-commodity flow network design problem with multiple concurrent edge failures. They formulate the problem as a two-stage RO model and solve it with a C&CG algorithm. Pishvaee et al. (2011) study a closed-loop supply chain network design problem, where box uncertainty sets are used to describe the randomness in demand, returns, and transportation costs. Zetina et al. (2017) consider both uncertain demand and transportation costs in uncapacitated hub location problems. They use budgeted uncertainty sets to characterize both uncertainties and the duality technique to reformulate the static robust models.

In terms of facility disruptions, Yu et al. (2017) study the uncapacitated fixed-charge location problem (UFLP) in a stochastic optimization framework by incorporating risk preferences. They propose conditional value-at-risk- and absolute semideviation- based models to control the risk of transportation cost at each customer. Afify et al. (2019) study a reliable p-median problem (PMP) and a reliable UFLP, where each facility has a heterogeneous failure probability and each customer is allocated to a primary facility and a back-up facility. They propose an evolutionary learning algorithm to solve the problem. Xie et al. (2019) study the reliable UFLP with correlated disruptions, which are solved by Lagrangian relaxation based algorithms. An et al. (2014) study a reliable PMP. Cheng et al. (2018b) consider a three-echelon logistics network design problem. Both studies use the two-stage RO framework and solve models using C&CG algorithms. For more details on reliable facility location, see the review paper by Snyder et al. (2016).

Baghalian et al. (2013) study a supply chain network design problem, considering supply-side and demand-side uncertainties simultaneously. They assume demand variables follow a known distribution function and describe supply-side uncertainty (facility disruption and link failure) through scenarios. Zokaee et al. (2016) study a humanitarian network design problem, which includes suppliers, relief distribution centers (RDC), and affected areas. Installation costs of RDC, shortage costs at affected areas, transportation costs at both echelons, supply capacity, and demand are subject to uncertainties. They use budgeted uncertainty sets to describe randomnesses and a static RO method to model the problem.

Table 1: Summary of literature review

	Type of uncertainty			Modeling scheme		
Authors	Provider	Receiver	In-between	Static	Two(multi)-stage	Solution method
Baron et al. (2011)		✓		RO		Duality technique
Atamtürk and Zhang (2007)		✓			RO	Cutting plane
Zeng and Zhao (2013)		✓			RO	Column-and-constraint generation
Ardestani-Jaafari and Delage (2017)		✓			RO	Affine policy
Gao and Qin (2016)			✓	CCP		Genetic algorithm
Mišković et al. (2017)			✓	RO		Memetic algorithm
Matthews et al. (2019)			✓		RO	Column-and-constraint generation
Pishvaee et al. (2011)		✓	✓	RO		Duality technique
Zetina et al. (2017)		✓	✓	RO		Duality technique
Yu et al. (2017)	✓			Stochastic		Branch-and-cut, Lagrangian decomposition
Afify et al. (2019)	✓			Stochastic		Evolutionary learning algorithm
Xie et al. (2019)	✓			Stochastic		Lagrangian relaxation
An et al. (2014)	✓				RO	Column-and-constraint generation
Cheng et al. (2018b)	✓				RO	Column-and-constraint generation
Baghalian et al. (2013)	✓	✓		Stochastic		Mixed-integer linear programming
Zokaee et al. (2016)	✓	✓	✓	RO		Duality technique
This paper	✓	✓			RO	Column-and-constraint generation, affine policy

From the literature, we can see that most works consider one type of uncertainty at a time. Although some papers consider multiple types of uncertainties simultaneously, their modeling schemes may produce overly conservative solutions as all decisions are made '*here-and-now*' (Pishvaee et al. 2011; Zokaee et al. 2016), or because it is impossible to enumerate all the disruption scenarios (Baghalian et al. 2013). Thus, this paper uses a two-stage RO method for the CFLP under uncertain demand and facility disruptions, which applies revealed uncertainty information to make recourse decisions, in order to generate less conservative solutions. Moreover, the two-stage method does not depend on probability distribution or scenario generation.

3 Models

Notation. We denote $\mathbb{R}$ as the space of real numbers, $\mathbb{R}_+$ as the space of positive real numbers, and $\mathbb{B}$ as the space of binary numbers. $|I|$ is the cardinality of set I. Let I and J be the sets of customers and facilities, respectively. The parameter f_j is the fixed cost of locating a facility at candidate site $j \in J$, and C_j is the corresponding capacity if we build a facility there. The parameter h_i is the demand quantity at customer $i \in I$, and d_{ij} is the cost for facility j to satisfy one unit of demand at customer $i \in I$. The unit penalty cost associated with unmet demand at customer $i \in I$ is p_i. We use $y_j = 1$ to denote that a facility is built at site $j \in J$, and $y_j = 0$ otherwise. The variable x_{ij} is the product quantity delivered from facility $j \in J$ to customer $i \in I$, and u_i is the unsatisfied demand at customer $i \in I$.

3.1 The deterministic model

The deterministic CFLP can be formulated as

$$
\begin{aligned}
\underline{\text{CFLP}}: \quad & \min \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij} + \sum_{i\in I} p_i u_i && && \text{(1a)}\\
& \text{s.t.} \sum_{j\in J} x_{ij} + u_i \geq h_i && \forall i \in I, && \text{(1b)}\\
& \sum_{i\in I} x_{ij} \leq C_j y_j && \forall j \in J, && \text{(1c)}\\
& y_j \in \{0,1\} && \forall j \in J, && \text{(1d)}\\
& x_{ij} \geq 0 && \forall i \in I, j \in J, && \text{(1e)}\\
& u_i \geq 0 && \forall i \in I. && \text{(1f)}
\end{aligned}
$$

Objective function (1a) minimizes the total cost, which involves the facility location cost, transportation cost, and the penalty cost of unsatisfied demand. Constraints (1b) denote that the sum of met and unmet demand must be greater than or equal to a customer's demand. Inequalities (1c) impose that customers can only be allocated to opened facilities and that a facility's capacity constraint must be respected. Constraints (1d)–(1f) impose the integrality and non-negativity constraints.

3.2 The robust model under uncertain demand and facility disruptions

We use a budgeted uncertainty set to characterize uncertain demand (Zeng and Zhao 2013; Bertsimas and Shtern 2018):

$$
\mathcal{U}_h = \left\{ \boldsymbol{h} \in \mathbb{R}_+^{|I|} : h_i = \bar{h}_i + \theta_i h_i^{\Delta}, 0 \leq \theta_i \leq 1, \sum_{i\in I} \theta_i \leq \Gamma_h \right\}, \tag{2}
$$

where $\bar{h}_i$ is the nominal (or basic) demand at customer i and $h_i^{\Delta} \geq 0$ is the maximal demand deviation. Γ_h is the uncertainty budget which bounds the maximal number of demand parameters by which values are allowed to deviate from their nominal values.

We characterize disruption risks as (An et al. 2014; Cheng et al. 2018b)

$$\mathcal{Z}_k = \left\{ \boldsymbol{z} \in \mathbb{B}^{|J|} : \sum_{j \in J} z_j \leq k \right\}, \tag{3}$$

where $z_j = 1$ if facility j is disrupted, and $z_j = 0$ otherwise. Equation (3) means that at most k facilities are allowed to fail simultaneously.

We use the following uncertainty set to represent simultaneous demand uncertainty and facility disruptions

$$\mathcal{W} = \left\{ (\boldsymbol{h}, \boldsymbol{z}) \in \mathbb{R}_+^{|I|} \times \mathbb{B}^{|J|} : \boldsymbol{h} \in \mathcal{U}_h, \boldsymbol{z} \in \mathcal{Z}_k \right\}. \tag{4}$$

The adjustable robust counterpart (ARC) model for CFLP is

$$\begin{aligned}
\underline{\text{CFLP-DR:}} \quad & \min_{\boldsymbol{y}, \boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)} \sup_{(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}} \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}(\boldsymbol{h}, \boldsymbol{z}) + \sum_{i \in I} p_i u_i(\boldsymbol{h}, \boldsymbol{z}) && \text{(5a)} \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij}(\boldsymbol{h}, \boldsymbol{z}) + u_i(\boldsymbol{h}, \boldsymbol{z}) \geq h_i \qquad \forall (\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}, i \in I, && \text{(5b)} \\
& \sum_{i \in I} x_{ij}(\boldsymbol{h}, \boldsymbol{z}) \leq C_j y_j (1 - z_j) \qquad \forall (\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}, j \in J, && \text{(5c)} \\
& y_j \in \{0, 1\} \qquad \forall j \in J, && \text{(5d)} \\
& x_{ij}(\boldsymbol{h}, \boldsymbol{z}) \geq 0 \qquad \forall (\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}, i \in I, j \in J, && \text{(5e)} \\
& u_i(\boldsymbol{h}, \boldsymbol{z}) \geq 0 \qquad \forall (\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}, i \in I. && \text{(5f)}
\end{aligned}$$

The objective function minimizes the worst-case cost. Here, we use $x_{ij}(\boldsymbol{h}, \boldsymbol{z}), \forall i \in I, j \in J$ and $u_i(\boldsymbol{h}, \boldsymbol{z}), \forall i \in I$ to indicate that we can delay the allocation decisions until we have observed customers' demand and facilities' status. Constraints (5c) mean that customers can only be reassigned to opened and functional facilities (i.e., those with $y = 1$ and $z = 0$). We use CFLP-D and CFLP-R to denote the model with only uncertain demand and with only facility disruptions, respectively. The two models can be obtained directly by setting the parameter $\Gamma_h = 0$ or $k = 0$. From model (5), we can observe that uncertainties only affect the right-hand side of constraints and that the model has the propriety of *fixed recourse* (i.e., the coefficients of recourse variables are not influenced by uncertainties).

4 Solution methods

Notation $x_{ij}(\boldsymbol{h}, \boldsymbol{z})$ and $u_i(\boldsymbol{h}, \boldsymbol{z})$ indicates that x_{ij} and u_i are no longer a single variable but rather a mapping from the space of observations $\mathbb{R}_+^{|I|} \times \mathbb{B}^{|J|}$ to $\mathbb{R}_+ \cup \{0\}$. This flexibility comes at the price of significant computational challenges. To solve the models, this section presents an exact algorithm based on a decomposition scheme and a heuristic algorithm based on the AP. We use the CFLP-DR to describe our algorithms.

4.1 Column-and-constraint generation

The C&CG algorithm is implemented in a master-subproblem framework. The master problem (MP) is solved to generate a first-stage solution, and the subproblem (SP) is solved to identify the worst-case realization of the uncertain parameters under a given first-stage solution. Each time after an SP is solved, we compute the gap between the upper and lower bounds. If the optimality gap is reached, the algorithm terminates; otherwise, we add the identified worst-case scenario and its associated variables and constraints to the MP, and the algorithm iterates.

Master problem. The MP is written as

$$
\begin{aligned}
\phi = & \min_{\boldsymbol{y}, s, \{\boldsymbol{x}\}_{l=1}^{n}, \{\boldsymbol{u}\}_{l=1}^{n}} s \\
\text{s.t. } & s \geq \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}^l + \sum_{i \in I} p_i u_i^l && \forall l \in \{1, \cdots, n\}, \\
& \sum_{j \in J} x_{ij}^l + u_i^l \geq h_i^l && \forall l \in \{1, \cdots, n\}, i \in I, \\
& \sum_{i \in I} x_{ij}^l \leq C_j y_j (1 - z_j^l) && \forall l \in \{1, \cdots, n\}, j \in J, \\
& y_j \in \{0, 1\} && \forall j \in J, \\
& x_{ij}^l \geq 0 && \forall l \in \{1, \cdots, n\}, i \in I, j \in J, \\
& u_i^l \geq 0 && \forall l \in \{1, \cdots, n\}, i \in I.
\end{aligned}
$$

The MP seeks to find the best location decision in light of the set of significant scenarios identified in the subproblem. The allocation variables, x_{ij}^l and u_i^l, now feature an extra index l, which means that these variables are added after finishing the lth iteration. Similarly, parameters h_i^l and z_j^l are the worst-case realization of random variables h_i and z_j identified in the lth iteration.

Subproblem. Since unmet demand is associated with a penalty cost, the second-stage problem is always feasible. Here, we use Karush–Kuhn–Tucker (KKT) conditions to derive the SP. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\lambda}$ be the dual variables associated with constraints (5b)–(5c) and (5e)–(5f), respectively. The SP is

$$
\begin{aligned}
\psi = & \max_{\substack{\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\lambda}, \\ \boldsymbol{w}^{\alpha}, \boldsymbol{w}^{\beta}, \boldsymbol{w}^{\gamma}, \boldsymbol{w}^{\lambda}, \boldsymbol{z}}} \sum_{j \in J} f_j \hat{y}_j + \sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij} + \sum_{i \in I} p_i u_i \\
\text{s.t. } & \sum_{j \in J} x_{ij} + u_i \geq h_i && \forall i \in I, \\
& \sum_{i \in I} x_{ij} \leq C_j \hat{y}_j (1 - z_j) && \forall j \in J, \\
& x_{ij} \geq 0 && \forall i \in I, j \in J, \\
& u_i \geq 0 && \forall i \in I, \\
& \alpha_i - \beta_j + \gamma_{ij} = d_{ij} && \forall i \in I, j \in J, \\
& \alpha_i + \lambda_i = p_i && \forall i \in I, \\
& \sum_{j \in J} x_{ij} + u_i \leq h_i + M_i^{\alpha} (1 - w_i^{\alpha}) && \forall i \in I, \\
& \alpha_i \leq M_i^{\alpha} w_i^{\alpha} && \forall i \in I, \\
& \sum_{i \in I} x_{ij} \geq C_j \hat{y}_j (1 - z_j) + M_j^{\beta} (w_j^{\beta} - 1) && \forall j \in J, \\
& \beta_j \leq M_j^{\beta} w_j^{\beta} && \forall j \in J, \\
& x_{ij} \leq M_{ij}^{\gamma} (1 - w_{ij}^{\gamma}) && \forall i \in I, j \in J, \\
& \gamma_{ij} \leq M_{ij}^{\gamma} w_{ij}^{\gamma} && \forall i \in I, j \in J, \\
& u_i \leq M_i^{\lambda} (1 - w_i^{\lambda}) && \forall i \in I, \\
& \lambda_i \leq M_i^{\lambda} w_i^{\lambda} && \forall i \in I, \\
& h_i = \bar{h}_i + \theta_i h_i^{\Delta} && \forall i \in I, \\
& 0 \leq \theta_i \leq 1 && \forall i \in I,
\end{aligned}
$$

$$\sum_{i \in I} \theta_i \leq \Gamma_h,$$
$$\sum_{j \in J} z_j \leq k,$$
$$x_{ij}, u_i, \alpha_i, \beta_j, \gamma_{ij}, \lambda_i \geq 0 \qquad \forall i \in I, j \in J,$$
$$w_i^\alpha, w_j^\beta, w_{ij}^\gamma, w_i^\lambda, z_j \in \{0,1\} \qquad \forall i \in I, j \in J.$$

Let $M_i^\alpha = p_i, M_j^\beta = \max\{C_j, \max_i\{d_{ij}(\bar{h}_i + h_i^\Delta), p_i(\bar{h}_i + h_i^\Delta)\}\}, M_{ij}^\gamma = \max\{C_j, d_{ij}(\bar{h}_i + h_i^\Delta), p_i(\bar{h}_i + h_i^\Delta)\}, M_i^\lambda = \max\{p_i, \bar{h}_i + h_i^\Delta\}$.

The detailed implementation of the C&CG algorithm is given in Algorithm 1. In Step 1, we solve the deterministic model and get an initial location decision $\hat{\boldsymbol{y}}$. In Step 2, we solve the SP with provided $\hat{\boldsymbol{y}}$ to identify the worst-case scenario and update the lower bound. If the termination condition is satisfied, the algorithm ends, else iteration continues. In Step 3, MP and SP are alternately solved to close the optimality gap.

Algorithm 1: C&CG algorithm for ARC model

Initialization: Let $LB = -\infty$, $UB = \infty, n = 0$.

Step 1: Solve the deterministic model with $h_i = \bar{h}_i$ to get an initial location decision $\hat{\boldsymbol{y}}$. Set LB as the objective value of the deterministic model.

Step 2: Solve the SP based on $\hat{\boldsymbol{y}}$ to find the worst-case scenario $(\hat{\boldsymbol{h}}, \hat{\boldsymbol{z}})$. Let $\hat{\psi}$ be the SP's optimal value. Set $UB = \min\{UB, \hat{\psi}\}$ and $n = n + 1$. If $(UB - LB)/UB \leq \epsilon$, the algorithm terminates; else, add the identified worst-case scenario and its associated variables and constraints to the MP and go to Step 3.

Step 3: Iterate until the algorithm terminates:

Step 3.1: Solve the MP to get a location decision $\hat{\boldsymbol{y}}$ and its optimal value $\hat{\phi}$. Set $LB = \hat{\phi}$.

Step 3.2: Solve the SP to identify the worst-case scenario and its optimal value $\hat{\psi}$. Set $UB = \min\{UB, \hat{\psi}\}$ and $n = n + 1$.

Step 3.3: If $(UB - LB)/UB \leq \epsilon$, the algorithm terminates; else, add the identified worst-case scenario and its associated variables and constraints to the MP, and go to Step 3.1.

4.2 Affine policy

Although the exact method described in the previous section is valuable, one cannot guarantee that it will converge in a reasonable amount of time. In most cases the ARC model is computationally intractable (Ben-Tal et al. 2004). This issue calls for some approximation schemes, among which the affine policy, or linear decision rule (LDR), is widely used. The LDR restricts adjustable variables to be affine functions of uncertain parameters. This leads to tractable robust counterparts for linear optimization problems with fixed recourse. In uncertainty set (4), since random variables $\boldsymbol{z}$ are independent from $\boldsymbol{\theta}$ and $\boldsymbol{z}$ can be relaxed to continuous variables (Cheng et al. 2018a), we are able to use the LDR for the robust model (5). Specifically, we approximate

$$\begin{cases} x_{ij} = \sum_{e \in I} A_{ije} h_e + \sum_{t \in J} B_{ijt} z_t + D_{ij} & \forall i \in I, j \in J, \\ u_i = \sum_{e \in I} E_{ie} h_e + \sum_{t \in J} F_{it} z_t + G_i & \forall i \in I, \end{cases} \tag{6}$$

where $A_{ije}, B_{ijt}, D_{ij}, E_{ie}, F_{it}$, and $G_i \in \mathbb{R}$. The resulting affinely adjustable robust counterpart (AARC) model for the CFLP-DR is

$$\min_{\substack{\boldsymbol{y}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}, \\ \boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}}} \sup_{(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}} \sum_{j \in J} f_j y_j + \sum_{i \in I} \sum_{j \in J} d_{ij} (\sum_{e \in I} A_{ije} h_e + \sum_{t \in J} B_{ijt} z_t + D_{ij})$$

$$+\sum_{i\in I}p_i(\sum_{e\in I}E_{ie}h_e+\sum_{t\in J}F_{it}z_t+G_i), \tag{7a}$$

$$\text{s.t.} \sum_{j\in J}(\sum_{e\in I}A_{ije}h_e+\sum_{t\in J}B_{ijt}z_t+D_{ij})+\sum_{e\in I}E_{ie}h_e+\sum_{t\in J}F_{it}z_t+G_i\geq h_i \quad \forall(\boldsymbol{h},\boldsymbol{z})\in\mathcal{W}, i\in I, \tag{7b}$$

$$\sum_{i\in I}(\sum_{e\in I}A_{ije}h_e+\sum_{t\in J}B_{ijt}z_t+D_{ij})\leq C_jy_j(1-z_j) \quad \forall(\boldsymbol{h},\boldsymbol{z})\in\mathcal{W}, j\in J, \tag{7c}$$

$$y_j\in\{0,1\} \quad \forall j\in J, \tag{7d}$$

$$\sum_{e\in I}A_{ije}h_e+\sum_{t\in J}B_{ijt}z_t+D_{ij}\geq 0 \quad \forall(\boldsymbol{h},\boldsymbol{z})\in\mathcal{W}, i\in I, j\in J, \tag{7e}$$

$$\sum_{e\in I}E_{ie}h_e+\sum_{t\in J}F_{it}z_t+G_i\geq 0 \quad \forall(\boldsymbol{h},\boldsymbol{z})\in\mathcal{W}, i\in I. \tag{7f}$$

To solve the AARC model: (i) We can develop an RG algorithm for an equivalent reformulation of model (7), based on the idea in Ardestani-Jaafari and Delage (2017). The resulting master and subproblems are linear programming models, and the master problem can be further strengthened by adding valid inequalities. (ii) We can first rewrite model (7) as an epigraph form, and then reformulate each robust constraint by applying duality theory (Gorissen et al. 2015). This method ultimately produces a mixed-integer linear programming (MILP) model. In the following sections, we implement both methods.

4.2.1 Row generation algorithm for the AARC model

In this section, we first derive an equivalent formulation for the AARC model and then develop an RG algorithm based on the equivalence.

Theorem 1. *The AARC model (7) is equivalent to*

$$\begin{aligned}\min_{\boldsymbol{y},\Psi}&\sum_{j\in J}f_jy_j+\Psi\\ s.t.\ &\Psi\geq g(\boldsymbol{y}),\\ &y_j\in\{0,1\} \qquad \forall j\in J,\end{aligned}$$

where $g(\boldsymbol{y})$ is defined as

$$\begin{aligned}\max_{\substack{\boldsymbol{W},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{A}',\boldsymbol{B}',\boldsymbol{E}',\\ \boldsymbol{F}',\boldsymbol{G}',\boldsymbol{S}',\boldsymbol{H}',\boldsymbol{L}',\boldsymbol{N}'}} &\sum_{i\in I}\bar{h}_iW_i+\sum_{i\in I}\sum_{\substack{e\in I,\\ e=i}}h_i^{\Delta}Y_{ie}-\sum_{j\in J}C_jy_jA'_j+\sum_{t\in J}\sum_{\substack{j\in J,\\ j=t}}C_jy_jE'_{tj}\\ s.t.\ &h_e^{\Delta}(Y_{ie}-B'_{ej}+G'_{ije}-d_{ij}\theta_e)=0 && \forall i\in I, j\in J, e\in I,\\ &Z_{it}-E'_{tj}+S'_{ijt}=d_{ij}z_t && \forall i\in I, j\in J, t\in J,\\ &W_i-A'_j+F'_{ij}=d_{ij} && \forall i\in I, j\in J,\\ &h_e^{\Delta}(Y_{ie}+L'_{ie}-p_i\theta_e)=0 && \forall i\in I, e\in I,\\ &Z_{it}+N'_{it}=p_iz_t && \forall i\in I, t\in J,\\ &W_i+H'_i=p_i && \forall i\in I,\\ &-W_i+Y_{ie}\leq 0 && \forall i\in I, e\in I,\\ &-\Gamma_hW_i+Y_{ie}\leq 0 && \forall i\in I, e\in I,\\ &-kW_i+\sum_{t\in J}Z_{it}\leq 0 && \forall i\in I,\\ &-A'_j+B'_{ej}\leq 0 && \forall e\in I, j\in J,\end{aligned}$$

$$
\begin{aligned}
& -\Gamma_h A'_j + \sum_{e\in I} B'_{ej} \leq 0 && \forall j \in J, \\
& -A'_j + E'_{tj} \leq 0 && \forall t \in J, j \in J, \\
& -kA'_j + \sum_{t\in J} E'_{tj} \leq 0 && \forall j \in J \\
& -F'_{ij} + G'_{ije} \leq 0 && \forall i \in I, j \in J, e \in I, \\
& -\Gamma_h F'_{ij} + \sum_{e\in I} G'_{ije} \leq 0 && \forall i \in I, j \in J, \\
& -F'_{ij} + S'_{ijt} \leq 0 && \forall i \in I, j \in J, t \in J, \\
& -kF'_{ij} + \sum_{t\in J} S'_{ijt} \leq 0 && \forall i \in I, j \in J, \\
& -H'_i + L'_{ie} \leq 0 && \forall i \in I, e \in I, \\
& -\Gamma_h H'_i + \sum_{e\in I} L'_{ie} \leq 0 && \forall i \in I, \\
& -H'_i + N'_{it} \leq 0 && \forall i \in I, t \in J, \\
& -kH'_i + \sum_{t\in J} N'_{it} \leq 0 && \forall i \in I, \\
& 0 \leq \theta_i \leq 1 && \forall i \in I, \\
& 0 \leq z_j \leq 1 && \forall j \in J, \\
& \sum_{i\in I} \theta_i \leq \Gamma_h, \\
& \sum_{j\in J} z_j \leq k, \\
& W_i, Y_{ie}, Z_{it}, A'_j, B'_{ej}, E'_{tj}, F'_{ij}, G'_{ije}, S'_{ijt}, H'_i, L'_{ie}, N'_{it} \geq 0 && \forall i \in I, e \in I, j \in J, t \in J.
\end{aligned}
$$

Proof. Please see Appendix A. □

Algorithm 2: Row generation algorithm for AARC model

Initialization: Let $LB = -\infty$, $UB = \infty, n = 1$.

Step 1: Solve the deterministic model with $h_i = \bar{h}_i$ to get an initial location decision $\hat{\boldsymbol{y}}^{(1)}$.

Step 2: Solve the subproblem $g(\hat{\boldsymbol{y}}^{(n)})$ and obtain its optimal value Ψ^*. Set $\dot{\boldsymbol{W}}^{(n)}$, $\dot{\boldsymbol{Y}}^{(n)}$, $\dot{\boldsymbol{A}}'^{(n)}$, $\dot{\boldsymbol{E}}'^{(n)}$ to their respective values based on the optimal solution. Let $UB = \min(UB, \sum_{j\in J} f_j \hat{y}_j^{(n)} + \Psi^*)$.

Step 3: Set $n = n + 1$ and solve the following master problem:

$$\min \sum_{j\in J} f_j y_j + \Psi, \tag{8a}$$

$$\text{s.t. } \Psi \geq \sum_{i\in I} \bar{h}_i \dot{W}_i^{(l)} + \sum_{i\in I} \sum_{\substack{e\in I,\\ e=i}} h_i^{\Delta} \dot{Y}_{ie}^{(l)} - \sum_{j\in J} C_j y_j \dot{A}_j^{'(l)} + \sum_{t\in J} \sum_{\substack{j\in J,\\ j=t}} C_j y_j \dot{E}_{tj}^{'(l)} \ \forall l \in \{1, \dots, n-1\}, \tag{8b}$$

$$y_j \in \{0, 1\} \qquad \forall j \in J. \tag{8c}$$

Let $\hat{\boldsymbol{y}}^{(n)}$ and $\Psi^{(n)}$ be the optimal solution and value of the master problem, respectively. Set $LB = \sum_{j\in J} f_j y_j^{(n)} + \Psi^{(n)}$.

Step 4: If $(UB - LB)/UB \leq \epsilon$, then the algorithm terminates, return $\hat{\boldsymbol{y}}^{(n)}$ as its optimal solution; otherwise repeat from Step 2.

Based on Theorem 1, we use an RG algorithm, given in Algorithm 2, to solve the AARC model. The RG algorithm is implemented in a master-subproblem framework. The master problem is to find the best location decision and the subproblem is to search for the optimality cut. We can improve the convergence speed of Algorithm 2 by adding valid inequalities (Ardestani-Jaafari and Delage 2017). For a pair $(\boldsymbol{y}, \Psi)$ to be feasible for model (7), for any $\{(\boldsymbol{h}, \boldsymbol{z})^{(l)}\}_{l\in\Lambda} \subset \mathcal{W}$, there must exist an assignment for $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$, and $\boldsymbol{G}$ such that the following constraints are satisfied:

$$
\begin{aligned}
&\Psi \geq \sum_{i\in I}\sum_{j\in J} d_{ij}(\sum_{e\in I} A_{ije}h_e^{(l)} + \sum_{t\in J} B_{ijt}z_t^{(l)} + D_{ij}) + \sum_{i\in I} p_i(\sum_{e\in I} E_{ie}h_e^{(l)} + \sum_{t\in J} F_{it}z_t^{(l)} + G_i) && \forall l \in \Lambda,\\
&\sum_{j\in J}(\sum_{e\in I} A_{ije}h_e^{(l)} + \sum_{t\in J} B_{ijt}z_t^{(l)} + D_{ij}) + \sum_{e\in I} E_{ie}h_e^{(l)} + \sum_{t\in J} F_{it}z_t^{(l)} + G_i \geq h_i^{(l)} && \forall l \in \Lambda, i \in I,\\
&\sum_{i\in I}(\sum_{e\in I} A_{ije}h_e^{(l)} + \sum_{t\in J} B_{ijt}z_t^{(l)} + D_{ij}) \leq C_j y_j(1 - z_j^{(l)}) && \forall l \in \Lambda, j \in J,\\
&\sum_{e\in I} A_{ije}h_e^{(l)} + \sum_{t\in J} B_{ijt}z_t^{(l)} + D_{ij} \geq 0 && \forall l \in \Lambda, i \in I, j \in J,\\
&\sum_{e\in I} E_{ie}h_e^{(l)} + \sum_{t\in J} F_{it}z_t^{(l)} + G_i \geq 0 && \forall l \in \Lambda, i \in I.
\end{aligned}
$$

Therefore, we propose an enhanced master problem:

$$
\begin{aligned}
\min_{\boldsymbol{y},\Psi}\ & \sum_{j\in J} f_j y_j + \Psi\\
\text{s.t. } & \Psi \geq \sum_{i\in I}\sum_{j\in J} d_{ij}(\sum_{e\in I} A_{ije}h_e + \sum_{t\in J} B_{ijt}z_t + D_{ij}) + \sum_{i\in I} p_i(\sum_{e\in I} E_{ie}h_e + \sum_{t\in J} F_{it}z_t + G_i)\ \forall(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}'\\
& \sum_{j\in J}(\sum_{e\in I} A_{ije}h_e + \sum_{t\in J} B_{ijt}z_t + D_{ij}) + \sum_{e\in I} E_{ie}h_e + \sum_{t\in J} F_{it}z_t + G_i \geq h_i \qquad \forall(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}', i \in I,\\
& \sum_{i\in I}(\sum_{e\in I} A_{ije}h_e + \sum_{t\in J} B_{ijt}z_t + D_{ij}) \leq C_j y_j(1 - z_j) \qquad \forall(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}', j \in J,\\
& \sum_{e\in I} A_{ije}h_e + \sum_{t\in J} B_{ijt}z_t + D_{ij} \geq 0 \qquad \forall(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}', i \in I, j \in J,\\
& \sum_{e\in I} E_{ie}h_e + \sum_{t\in J} F_{it}z_t + G_i \geq 0 \qquad \forall(\boldsymbol{h}, \boldsymbol{z}) \in \mathcal{W}', i \in I,\\
& \text{and (8b)–(8c),}
\end{aligned}
$$

where $\mathcal{W}'$ is a finite set of some feasible realizations of $(\boldsymbol{h}, \boldsymbol{z})$. In our implementation, $\mathcal{W}'$ only includes the most recently identified worst-case scenario, as was the case in Ardestani-Jaafari and Delage (2017).

4.2.2 Reformulation of the AARC model

We can derive the robust counterpart of model (7) using duality theory, which produces a MILP model and it can be directly solved by an off-the-shelf solver. The reformulation is

$$
\min_{\substack{\boldsymbol{y},\boldsymbol{\eta},\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\nu},\boldsymbol{\rho},\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{E},\boldsymbol{F},\boldsymbol{G},\boldsymbol{S},\\ \boldsymbol{H},\boldsymbol{L},\boldsymbol{N},\boldsymbol{O},\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R},\boldsymbol{K},\boldsymbol{T},\boldsymbol{U},\boldsymbol{\zeta},\boldsymbol{\xi},\delta,\pi}} s \tag{9a}
$$

$$
\begin{aligned}
\text{s.t. } s \geq \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_{ij}D_{ij} + \sum_{i\in I} p_i G_i + \sum_{i\in I}\sum_{j\in J}\sum_{e\in I} d_{ij}A_{ije}\bar{h}_e + \sum_{i\in I}\sum_{e\in I} p_i E_{ie}\bar{h}_e&\\
+ \sum_{e\in I}\zeta_e + \Gamma_h\delta + \sum_{t\in J}\xi_t + k\pi,& \qquad (9b)
\end{aligned}
$$

$$
\zeta_e + \delta \geq \sum_{i\in I}\sum_{j\in J} d_{ij}A_{ije}h_e^{\Delta} + \sum_{i\in I} p_i E_{ie}h_e^{\Delta} \qquad \forall e \in I, \tag{9c}
$$

$$\xi_t + \pi \geq \sum_{i \in I} \sum_{j \in J} d_{ij} B_{ijt} + \sum_{i \in I} p_i F_{it} \qquad \forall t \in J, \tag{9d}$$

and (A.1)–(A.4). (9e)

4.3 Optimality of the affine policy

Although the AP is an approximate method, it can generate optimal first-stage solutions in some cases. Table 2 identifies conditions under which the AP is optimal for each robust model.

Table 2: Optimality of the linear decision rule

k	Γ_h	CFLP-D	CFLP-R	CFLP-DR
0	0	✓	✓	✓
	1	✓	Not applicable	✓
	$(1, \lvert I \rvert)$	×	Not applicable	×
	$\lvert I \rvert$	✓	Not applicable	✓
1	0	Not applicable	✓	✓
	1	Not applicable	Not applicable	×
	$(1, \lvert I \rvert)$	Not applicable	Not applicable	×
	$\lvert I \rvert$	Not applicable	Not applicable	✓
$(1, \lvert J \rvert)$	0	Not applicable	×	×
	$[1, \lvert I \rvert]$	Not applicable	Not applicable	×
$\lvert J \rvert$	0	Not applicable	✓	✓
	$[1, \lvert I \rvert]$	Not applicable	Not applicable	✓

From Table 2, we get the following conclusions:

1. *When $k = 0$ and $\Gamma_h = 0$, the robust models reduce to the deterministic model.*

2. *For the CFLP-D, when $\Gamma_h = 1$ and $\Gamma_h = |I|$, the AP gives optimal first-stage solutions.*
   • When $\Gamma_h = 1$, the uncertainty set $\mathcal{U}_h$ is a simplex, i.e., it is a convex combination of $(|I| + 1)$ affinely independent points in $\mathbb{R}_+^{|I|}$. These points are $(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_i)$, $(\bar{h}_1 + h_1^\Delta, \bar{h}_2, \ldots, \bar{h}_i)$, $(\bar{h}_1, \bar{h}_2 + h_2^\Delta, \ldots, \bar{h}_i)$,...,$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_i + h_i^\Delta)$. Based on Theorem 1 in Bertsimas and Goyal (2012), for a two-stage adaptive RO problem under right-hand side uncertainty with a min-max objective, the AP identifies optimal solutions when the uncertainty set is a simplex.
   • When $\Gamma_h = |I|$, constraint $\sum_{i \in I} \theta_i \leq \Gamma_h$ in Equation (2) is redundant as $0 \leq \theta_i \leq 1, \forall i \in I$. Since only constraints (2) involve uncertain parameter h_i at the right-hand side of the ARC model, in this situation the uncertainty becomes constraint-wise. According to Theorem 2.1 in Ben-Tal et al. (2004), the ARC is now equivalent to the static RO model. We can set $x_{ij} = D_{ij}, u_i = G_i, \forall i \in I, j \in J$ in Equation (6), and the resulting AARC model (7) is exactly the static RO model. This means that both the AARC model and the ARC model are now equivalent to the static RO model.

3. *For the CFLP-R, when $k = 1$ and $k = |J|$, the AP is optimal.*
   We first relax the uncertainty set (3) to its convex hull $\mathcal{Z}_k = \left\{ \boldsymbol{z} \in [0,1]^{|J|} : \sum_{j \in J} z_j \leq k \right\}$. Then the proof process is the same as that of the CFLP-D. Further, when $k = |J|$, both the ARC and AARC models generate solutions with no opened facilities as the static RO model does (proved in Appendix B).

4. *For the CFLP-DR, when $k = 0$ (or $\Gamma_h = 0$), it reduces to the CFLP-D (or CFLP-R). When $k = 1$ and $\Gamma_h = |I|$, the CFLP-DR is equivalent to the CFLP-R with $h_i = \bar{h}_i + h_i^\Delta, \forall i \in I$; therefore, the AP identifies optimal solutions for the CFLP-DR as it is optimal for the CFLP-R when $k = 1$. When $k = |J|$, the ARC and AARC models provide optimal solutions with no opened facilities, regardless of the value of Γ_h.*
   The latter two cases are proved in Appendix C.

5 Computational experiments

We adopt the instances generated by Cheng et al. (2018a) with slight modifications, which are originally from Daskin (2011). These instances are derived from 1990 census data. The 49 nodes include the state capitals of the continental United States and Washington, D.C. There are 35 instances in total. The nominal demand $\bar{h}_i = P_i \times 10^{-5}$, where P_i is the population at node i. We generate the maximal demand deviation h_i^{Δ} uniformly from the interval $[0.15\bar{h}, \bar{h}]$. The transportation cost d_{ij} is the great circle distance between nodes i and j in miles. For simplicity, we set the unit penalty cost p_i the same for all the customers, which is the greatest travel distance in the system. We denote instances as *Fac-X-Cus-Y*, which means that the considered instance has X candidate facilities and Y customers.

All the algorithms and models were coded in Python programming language, using Gurobi 8.1.1 as the solver. The calculations were run on a cluster of Lenovo SD350 servers with 2.4 GHz Intel Skylake cores and 202 GB of memory under Linux CentOS 7 system. Each experiment was conducted on a four-core processor of one node.

5.1 The impact of uncertainty on optimal solutions

We use instances Fac-10-Cus-10 and Fac-15-Cus-15 to conduct the experiments and set $\Gamma_h = 0.2|I|$ and $k = 2$. Results are presented in Table 3. We set the deterministic model's results as benchmarks and the other models' results are normalized by dividing those of the deterministic model. A ratio smaller (or larger) than 1 means that the robust models generate solutions of smaller (or larger) costs. Note that the nominal costs of the robust models are calculated by fixing the location decisions and solving resulting minimum cost flow problems. The worst-case cost of the deterministic model is obtained by fixing the location decision and solving the subproblem of the C&CG algorithm.

Table 3: The impact of uncertainty on location decision and cost

Instance	Model	Opened facilities	Nominal cost ratio	Worst-case cost ratio
Fac-10-Cus-10	CFLP	[0, 2, 3, 6, 8]*	1.00	1.00
	CFLP-D	[0, 2, 3, 5, 6, 8]	1.04	0.94
	CFLP-R	[0, 2, 3, 4, 5, 6, 8]	1.08	0.68
	CFLP-DR	[0, 2, 3, 4, 5, 6, 7, 8]	1.13	0.73
Fac-15-Cus-15	CFLP	[0, 1, 2, 3, 4, 5, 7]	1.00	1.00
	CFLP-D	[0, 1, 2, 3, 4, 5, 7]	1.00	1.00
	CFLP-R	[0, 1, 2, 3, 4, 5, 7, 14]	1.09	0.94
	CFLP-DR	[0, 1, 2, 3, 4, 5, 7, 10, 14]	1.17	0.82

* Facilities are indexed from 0.

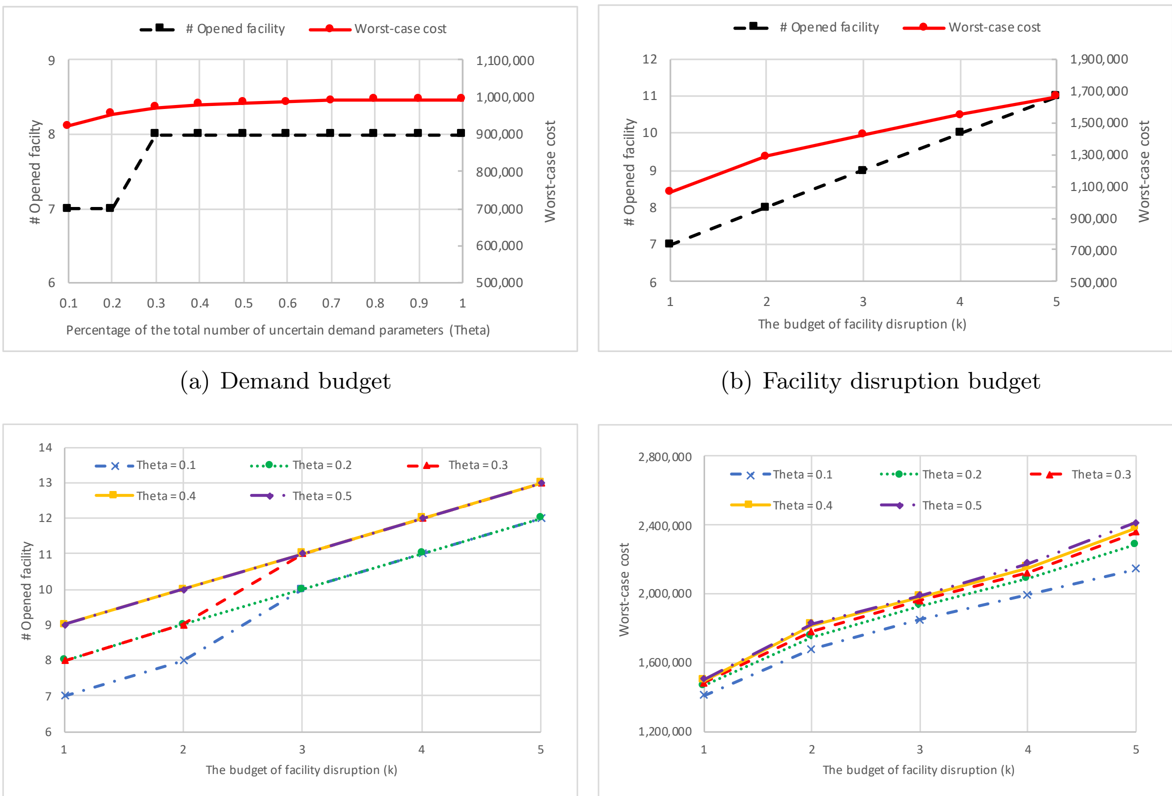
The first impact is the selection of opened facilities. As expected, when uncertainties are considered, more facilities are opened to mitigate potential risks. The CFLP-DR model generates solutions of the greatest number of opened facilities, due to the fact that two types of uncertainties are considered simultaneously. The second impact is cost. Generally, considering uncertainty increases the nominal cost and decreases the worst-case cost. Table 3 shows that facility disruption risk has a greater influence on location and cost, compared to demand uncertainty. For instance Fac-10-Cus-10, the CFLP-D opens one more facility than the deterministic model. However, when disruption risk is further considered, the CFLP-DR locates two more facilities compared to the CFLP-D. For instance Fac-15-Cus-15, the location decision of the CFLP-D is the same as the deterministic problem. However, the CFLP-R and CFLP-DR generate different solutions with more opened facilities.

Table 3 also suggests that sometimes a slight increase in the nominal cost can lead to a significant decrease in the worst-case cost. For example, for instance Fac-10-Cus-10, the nominal cost ratio of the CFLP-R (CFLP-DR) is 1.08 (1.13) whereas the worst-case cost ratio is 0.68 (0.73). This observation is consistent with other works that study reliable facility location problems (Snyder and Daskin 2005; An et al. 2014).

5.2 The impact of uncertainty budget on optimal solutions

We denote $\Gamma_h = \Theta|I|$, which means there are at most Θ of customers whose demand parameters are allowed to deviate from their nominal values. For space consideration, we only present the results of instance Fac-15-Cust-15 in Figure 1.

Figure 1(a) indicates that budget of demand uncertainty has a slight impact on the location decision and the worst-case cost. When Θ increases from 0.2 to 0.3, one more facility is opened. When $\Theta \geq 0.3$, the number of opened facilities stays the same, and the curve of the worst-case cost is relatively flat. On the contrary, the budget of facility disruption has a significant influence on facility configuration and the worst-case cost. Figure 1(b) displays that the number of opened facilities and the worst-case cost increase almost linearly with respect to k for the CFLP-R.



(a) Demand budget

(b) Facility disruption budget

(c) Demand and facility disruption budgets

(d) Demand and facility disruption budgets

Figure 1: The impact of uncertainty budget (Instance Fac-15-Cus-15)

Figures 1(c)–1(d) are the summarized results of the CFLP-DR. The detailed results are given in Table 4, where the last column is obtained by setting the results in the first row of different Θ values as benchmarks. Both figures show that the uncertainty of facility disruption plays a dominating role. Under the same value of k, the difference in the number of opened facilities is not significant, especially when $k \geq 3$, and the worst-case costs are also close. However, under the same value of Θ, the number of opened facilities and the worst-case cost ratio increase almost linearly with k. We note that sometimes even the number of opened facilities is the same under the same value of k for different Θ, there might be a difference in facility configuration. For example, according to Table 4, when $k = 4$, 11 sites are opened when $\Theta = 0.1$ and $\Theta = 0.2$. However, facilities 6 and 8 are opened in the former circumstance, and facilities 1 and 11 are opened in the latter.

5.3 Algorithm evaluation

This section evaluates the solution methods. The optimality tolerance for both C&CG and RG algorithms is set to 10^{-4}. The time limit is 7200 seconds (when this limit is reached, we still allow the

Table 4: Detailed results of the CFLP-RD for Instance Fac-15-Cust-15

Θ	k	Opened facilities	# Opened facilities	Worst-case cost ratio
0.1	1	[0, 1, 2, 3, 4, 5, 7]	7	1.00
	2	[0, 1, 2, 4, 5, 7, 10, 14]	8	1.19
	3	[0, 1, 2, 4, 5, 7, 10, 11, 13, 14]	10	1.31
	4	[0, 2, 3, 4, 5, **6**, 7, **8**, 10, 13, 14]	11	1.41
	5	[0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.52
0.2	1	[0, 1, 2, 3, 4, 5, 7, 14]	8	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 14]	9	1.20
	3	[0, 1, 2, 4, 5, 7, 10, 11, 13, 14]	10	1.32
	4	[0, **1**, 2, 3, 4, 5, 7, 10, **11**, 13, 14]	11	1.42
	5	[0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.56
0.3	1	[0, 1, 2, 3, 4, 5, 7, 14]	8	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 14]	9	1.20
	3	[0, 2, 3, 4, 5, 7, 8, 10, **11**, 13, 14]	11	1.32
	4	[0, 2, 3, 4, 5, 6, 7, **8**, 10, 11, 13, 14]	12	1.43
	5	[0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14]	13	1.59
0.4	1	[0, 1, 2, 3, 4, 5, 7, 11, 14]	9	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 11, 14]	10	1.21
	3	[0, **1**, 2, 3, 4, 5, 7, 8, 10, 13, 14]	11	1.32
	4	[0, **1**, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.44
	5	[0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14]	13	1.59
0.5	1	[0, 1, 2, 3, 4, 5, 7, 11, 14]	9	1.00
	2	[0, 1, 2, 3, 4, 5, 7, 10, 11, 14]	10	1.22
	3	[0, **1**, 2, 3, 4, 5, 7, 8, 10, 13, 14]	11	1.32
	4	[0, **1**, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14]	12	1.45
	5	[0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14]	13	1.61

current iteration to be completed). In following tables, *Ref* represents the reformulated model (9). *#Opt* is the number of instances that are solved to optimality. *#Iter* is the number of iterations. *Gap* is the optimality gap between the upper and lower bounds. *Bound gap* refers to the relative difference between the optimal worst-case cost f_C^* (generated by C&CG algorithm) and the worst-case bound f_L^* (the upper bound generated by the LDR). Mathematically, it is measured using $(f_L^* - f_C^*)/f_C^* \times 100\%$. *Opt gap* is the relative difference between f_C^* and the achieved worst-case cost f_A^*. The "achieved worst-case cost" of a location decision refers to the actual worst-case cost if the decision is applied. Mathematically, f_A^* is obtained by fixing the location decision generated by the LDR and solving the subproblem of C&CG, and *Opt gap* $= (f_A^* - f_C^*)/f_C^* \times 100\%$. Note that for the Opt gaps, we only make comparisons for instances that can be optimally solved by the C&CG algorithm. Meanwhile, we only report the Opt gaps for algorithms (used for the AARC models) which provide smaller bound gaps.

5.3.1 The CFLP-D

Table 5 and Figure 2 present the results of the CFLP-D. Table 5 shows that the C&CG algorithm can generate optimal solutions for almost all the instances with different budgets in a shorter time. As to the AARC model, the RG algorithm is more efficient than solving the reformulation directly. Specifically, the RG algorithm yields solutions with smaller bound gaps and consumes less computing time. The small bound and Opt gaps between the solutions of the C&CG algorithm and those of the RG algorithm suggest that the LDR can provide good approximation solutions for the CFLP-D.

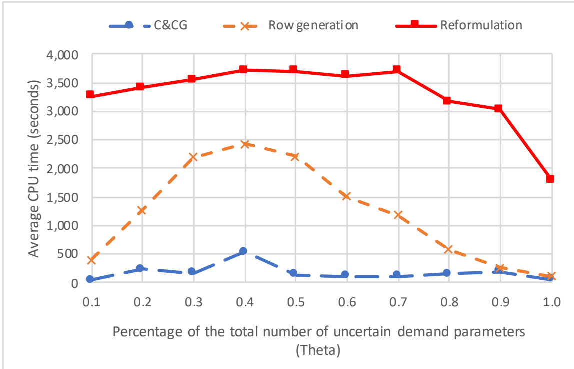
From Figure 2(a), the iteration number of the C&CG algorithm is small. It first increases and then decreases with the increasing uncertainty budget. When $\Theta = 1$, the C&CG algorithm finds optimal solutions in only 1 iteration. Figure 2(b) shows that the performance of the C&CG algorithm is not heavily affected by uncertainty budget. On the contrary, the computational efficiency of the RG algorithm is sensitive to the budget. For a small budget ($\Theta = 0.1$) or a large budget ($\Theta = 0.9$ or 1.0), the RG algorithm is relatively fast; however, in other cases, its CPU time has a wide variation. The reformulation model has seen an obvious decrease in CPU time when $\Theta \geq 0.7$; however, it always consumes the most time among the three solution methods.

Table 5: Algorithm comparison for the CFLP-D (average results)

					AARC model from the linear decision rule				
	C&CG				Row generation			Reformulation	
Θ	#Opt	#Iter	Gap	CPU	Bound gap	Opt gap	CPU	Bound gap	CPU
0.1	35/35*	2.77	0.00	48.72	0.02	0.24	400.58	4.14	3260.48
0.2	35/35	4.20	0.00	233.41	0.01	0.00	1281.14	7.06	3407.41
0.3	35/35	5.14	0.00	163.03	0.02	0.78	2194.35	39.83	3554.42
0.4	34/35	5.14	0.01	540.80	0.02	0.00	2433.38	12.14	3708.92
0.5	35/35	4.77	0.00	135.07	0.02	0.28	2214.24	14.98	3704.56
0.6	35/35	4.11	0.00	108.40	0.02	0.09	1515.89	14.78	3613.36
0.7	35/35	3.57	0.00	110.08	0.01	0.00	1177.02	3.10	3706.80
0.8	35/35	3.14	0.00	146.05	0.00	0.00	577.15	2.76	3170.10
0.9	35/35	2.57	0.00	190.17	0.00	0.00	261.71	3.64	3029.60
1.0	35/35	1.00	0.00	62.49	0.00	0.00	106.26	0.03	1790.31

* : The number of instances (out of 35) that are solved to optimality.

(a) #Iter of the C&CG algorithm



(b) CPU time

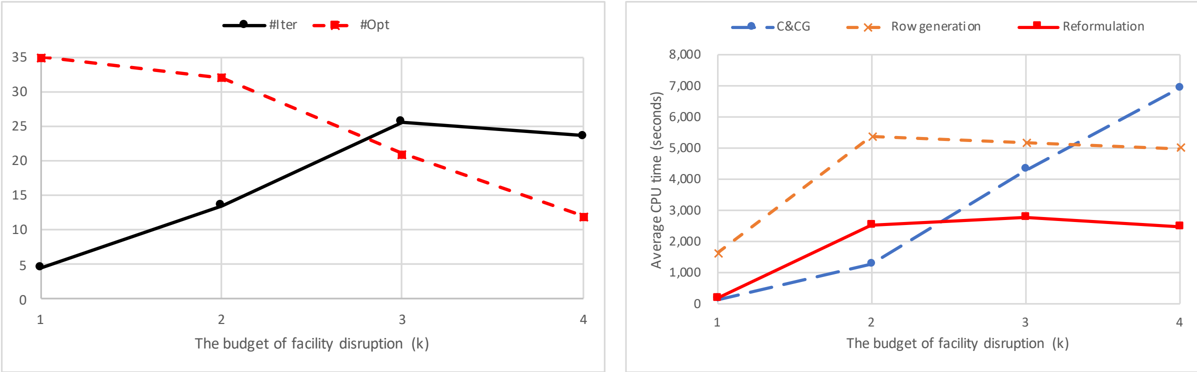
Figure 2: Average results of different algorithms for the CFLP-D

5.3.2 The CFLP-R

Table 6 presents the results of the CFLP-R. It displays that the C&CG algorithm can solve most instances to optimality for a small budget. Specifically, for $k = 1$ and $k = 2$, 35 and 32 (out of 35) instances can be optimally solved, respectively. However, Figure 3(a) demonstrates that this number decreases significantly with an increasing budget. The number of iterations increases over the budget k in general. With respect to the AARC model, the reformulation method provides solutions with smaller bound gaps in a shorter time, compared to the RG algorithm. When $k = 1$, the reformulation method identifies optimal first-stage solutions with Opt gaps being 0 as indicated in Section 4.3. Table 6 also suggests that even though the bound gaps are large, the Opt gaps are acceptable. Figure 3(b) shows that the CPU time of the C&CG algorithm increases almost linearly when $k \geq 2$; however, the computing time of the reformulation method is relatively stable and also shorter. Thus, for the CFLP-R with a large budget, we can consider using the LDR with the reformulation method to solve the robust model approximately.

Table 6: Algorithm comparison for the CFLP-R (average results)

					AARC model from the linear decision rule				
	C&CG				Row generation		Reformulation		
k	#Opt	#Iter	Gap	CPU	Bound gap	CPU	Bound gap	Opt gap	CPU
1	35/35	4.57	0.00	150.40	0.02	1606.99	0.00	0.00	164.97
2	32/35	13.54	1.37	1262.03	48.11	5372.04	46.53	7.91	2514.88
3	21/35	25.66	5.71	4285.34	46.72	5165.26	44.81	9.38	2775.00
4	12/35	23.63	14.31	6901.59	39.64	4987.98	37.44	5.05	2459.19



(a) #Opt and #Iter of the C&CG algorithm (b) CPU time

Figure 3: Average results of different algorithms for the CFLP-R

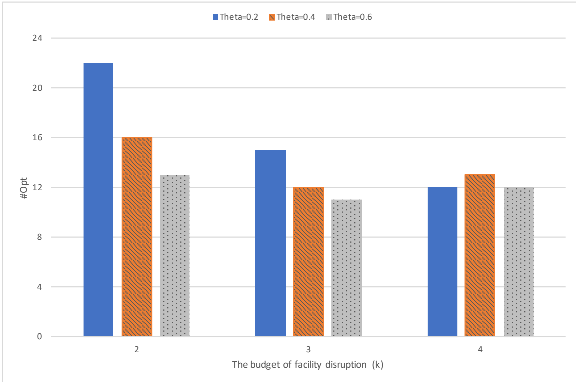
5.3.3 The CFLP-DR

Table 7 summarizes the results of the CFLP-DR. It shows that the computational efficiency of the C&CG algorithm is heavily affected when both uncertainties are simultaneously considered. Specifically, fewer instances are solved to optimality and the gaps are also larger compared to the CFLP-D and the CFLP-R. Figure 4(a) further shows that when k is small ($k = 2$), the #Opt has a large variation under different values of Θ. Similarly, when Θ is small ($\Theta = 0.2$), the #Opt also varies significantly with respect to k. For the AARC model, the RG algorithm provides solutions with smaller bound gaps than the reformulation method does. The Opt gaps are relatively large when $\Theta = 0.2$. And the RG algorithm provides solutions with better Opt gaps when $\Theta = 0.6$. Figure 4(c) shows that the CPU time of the C&CG algorithm has a lot of variation. However, the CPU time of the other two algorithms is quite stable. Although the reformulation method consumes less time, its solutions are inferior to those of the RG algorithm. From Figures 4(b)–4(c), when $\Theta = 0.2$, the C&CG algorithm has the largest iteration number while it consumes the least CPU time, compared to the cases of $\Theta = 0.4$ and $\Theta = 0.6$, which indicates that the computing time during each iteration is relatively shorter when $\Theta = 0.2$. In contrast, when $\Theta = 0.6$, the C&CG algorithm consumes the most CPU time with the smallest number of iterations.

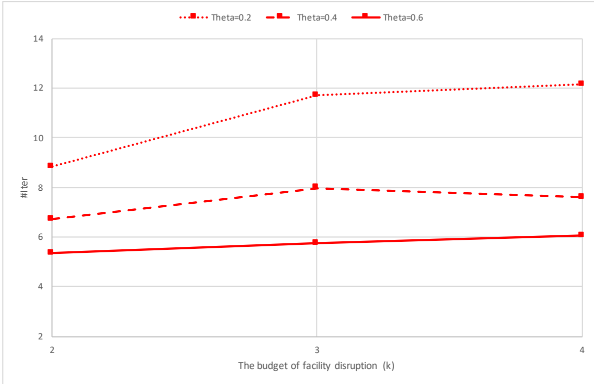
Table 7: Algorithm comparison for the CFLP-DR (average results)

						AARC model from the linear decision rule				
		C&CG				Row generation			Reformulation	
Θ	k	#Opt	#Iter	Gap	CPU	Bound gap	Opt gap	CPU	Bound gap	CPU
0.2	2	22/35	8.83	11.59	6141.89	47.85	10.21	5414.68	77.53	3217.34
	3	15/35	11.71	15.60	8043.41	50.04	11.57	5449.15	53.64	3337.77
	4	12/35	12.14	20.14	10932.31	42.55	9.46	5409.59	46.28	3251.51
0.4	2	16/35	6.71	16.74	8133.65	35.36	11.48	5358.68	42.46	3076.00
	3	12/35	7.97	19.53	9202.40	44.52	5.69	5363.89	45.00	3237.11
	4	13/35	7.60	22.99	11023.90	41.11	6.43	5191.64	48.05	3155.73
0.6	2	13/35	5.34	19.93	11307.78	27.59	4.89	5262.17	30.01	3194.63
	3	11/35	5.74	19.51	13019.47	39.75	7.00	5316.69	41.17	3223.54
	4	12/35	6.06	24.81	13239.51	37.42	2.33	4991.64	44.40	3247.58

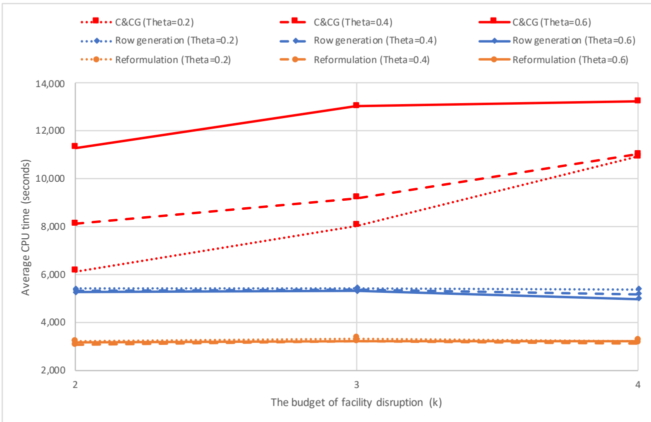
Conclusions. (1) For the CFLP-D, the C&CG algorithm is the most efficient one among the three algorithms. The LDR can provide good approximations with minor bound and Opt gaps. (2) For the CFLP-R, the efficiency of the C&CG algorithm is sensitive to the budget k. The CPU time of the reformulation model is relatively stable, and it is less than that of the C&CG when k is large. (3) The CFLP-DR is the most difficult one among the three robust models. The reformulation model resulting from the LDR can be used to get approximate solutions when both uncertainty budgets have large values.



(a) #Opt of the C&CG algorithm



(b) #Iter of the C&CG algorithm



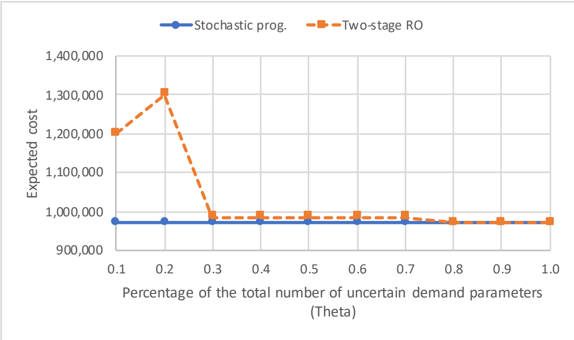
(c) CPU time

Figure 4: Average results of different algorithms for the CFLP-DR

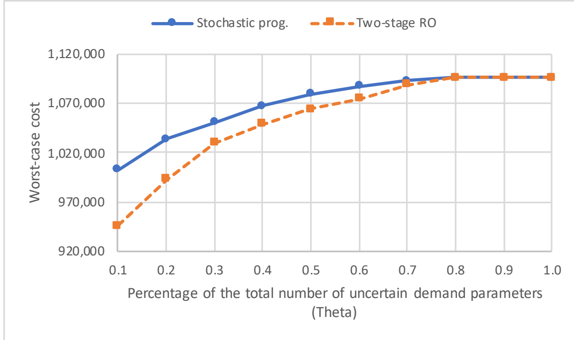
5.4 A comparison of robust and stochastic solutions

In this section, we compare the solutions obtained form the two-stage RO framework and the two-stage stochastic programming model presented in Appendix D. For the stochastic model, we sample 100 scenarios from independent and uniformly distributed demand $h_i \sim U(\bar{h}_i, 2\bar{h}_i), \forall i \in I$. For each scenario, we also incorporate potential facility disruptions. Specifically, we first generate a random float number $r \in [0, 1]$ for scenario $s \in S$. If $r \leq 0.5$, no facility is disrupted in scenario s; otherwise, we randomly generate two integer numbers in $[0, |J| - 1]$ (facilities are indexed from 0) to represent that the corresponding facilities are disrupted. Note that the two numbers can be the same, which means that only one facility fails. We set the occurrence probability of scenario s to $o_s = 1/100$. For the stochastic model with only facility disruptions, customer demand is set to the nominal value $\bar{h}_i$. Experiments are conducted using instance Fac-10-Cust-15. Figure 5 plots the results. The expected cost for the robust solution is obtained by solving the stochastic model with a given robust location decision. The worst-case cost for the stochastic solution is obtained by solving the subproblem of the C&CG algorithm. Note that we do not report the results of the static RO models, because they generate very conservative solutions when disruption risks are considered, i.e., no facility is opened and all customer demand will be lost.

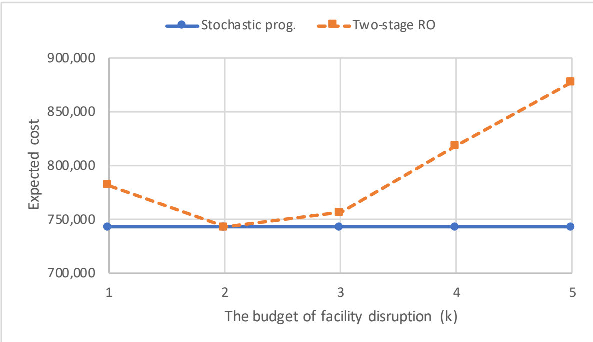
Figures 5(a) and 5(e) show that for a large value of Θ, the expected costs realized by the robust models are close to the minimum generated by the stochastic models. This indicates that even though the RO models do not aim to minimize the expected cost, we can adjust the uncertainty budget to obtain reasonable solutions with regard to the expectation criterion. Note that in the samples, customer demand has a large variation, i.e., between $\bar{h}_i$ and $2\bar{h}_i$, so a large uncertainty budget produces good expectation solutions here. Correspondingly, if customer demand is generated between a small interval, we can adjust Θ to small values to satisfy the expectation criterion. Figure 5(c) indicates



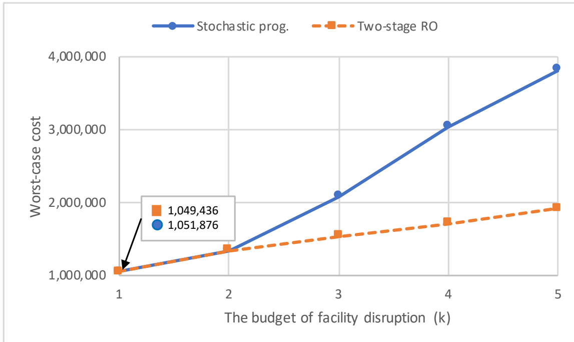
(a) Demand uncertainty: Expected cost



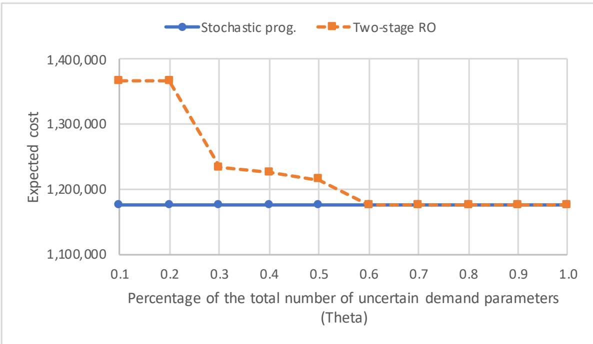
(b) Demand uncertainty: Worst-case cost



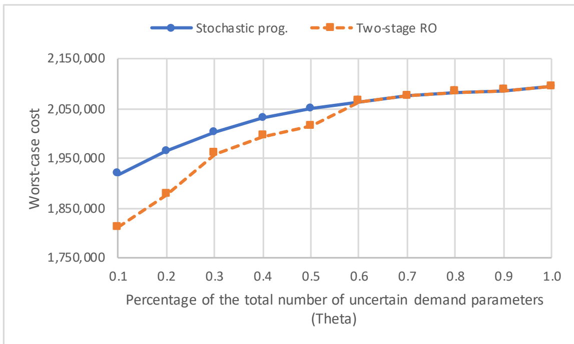
(c) Facility disruption: Expected cost



(d) Facility disruption: Worst-case cost



(e) Demand & facility uncertainty: Expected cost



(f) Demand & facility uncertainty: Worst-case cost

Figure 5: Comparison of the two-stage robust and stochastic solutions (Instance Fac-10-Cust-15)

that the robust and the stochastic models realize the same expected cost when $k = 2$. For $k = 1$ and 3, the expected costs of robust solutions are slightly higher. However, when $k = 4$ and 5, the differences become large. This is because when generating the samples, we allow at most 2 facilities to be disrupted in each scenario. Therefore, a large uncertainty budget means a large deviation from the samples, leading to poor performance in terms of expectation criterion. Figures 5(b), 5(d), and 5(f) suggest that in terms of worst-case cost, the performance of the robust models is better than or as good as that of the stochastic models.

6 Conclusions

This paper solves a fixed-charge location problem where two categories of parameters are subject to uncertainties simultaneously: demand and facility availability. We apply a two-stage RO framework for the problem, which allows allocation decisions to be made after the uncertainties are realized. We develop a C&CG algorithm and use the LDR to solve the model. We identify conditions under which the LDR produces optimal first-stage solutions. Numerical tests indicate that disruption risk has a greater effect on solution configuration and cost compared to demand uncertainty. For the CFLP-D, the C&CG algorithm is quite efficient and the LDR also produces good approximate solutions. For the CFLP-R and the CFLP-DR, the LDR can provide solutions with acceptable optimality gaps in a shorter time when the uncertainty budgets have large values. Numerical tests also demonstrate that we can adjust the uncertainty budget for robust models to generate reasonable solutions with respect to the expectation criterion.

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Appendix A Proof of Theorem 1

We first derive the robust counterpart of constraints (7b) as

$$\begin{aligned}
&\sum_{j\in J}\sum_{e\in I} A_{ije}\bar{h}_e + \sum_{e\in I} E_{ie}\bar{h}_e + \sum_{j\in J} D_{ij} + G_i - \bar{h}_i - \sum_{e\in I}\eta_{ie} - \Gamma_h\mu_i - \sum_{t\in J}\sigma_{it} - k\nu_i \geq 0 \ \forall i\in I,\\
&-\eta_{ie} - \mu_i \leq \sum_{j\in J} A_{ije}h_e^{\Delta} + E_{ie}h_e^{\Delta} && \forall i\in I, e\in I, i\neq e,\\
&-\eta_{ie} - \mu_i \leq \sum_{j\in J} A_{ije}h_e^{\Delta} + E_{ie}h_e^{\Delta} - h_i^{\Delta} && \forall i\in I, e\in I, i = e,\\
&\sigma_{it} - \nu_i \leq \sum_{j\in J} B_{ijt} + F_{it} && \forall i\in I, t\in J,\\
&\eta_{ie}, \mu_i, \sigma_{it}, \nu_i \geq 0, && \forall i\in I, e\in I, t\in J.
\end{aligned} \tag{A.1}$$

Similarly, we can get the robust counterpart of constraints (7c) as

$$\begin{aligned}
&\sum_{i\in I}\sum_{e\in I} A_{ije}\bar{h}_e + \sum_{i\in I} D_{ij} - C_j y_j + \sum_{e\in I}\rho_{ej} + \Gamma_h S_j + \sum_{t\in J} H_{tj} + kL_j \le 0 && \forall j \in J,\\
&\rho_{ej} + S_j \ge \sum_{i\in I} A_{ije} h_e^{\Delta} && \forall e \in I, j \in J,\\
&H_{tj} + L_j \ge \sum_{i\in I} B_{ijt} && \forall t \in J, j \in J, t \ne j\\
&H_{tj} + L_j \ge \sum_{i\in I} B_{ijt} + C_j y_j && \forall t \in J, j \in J, t = j,\\
&\rho_{ej}, S_j, H_{tj}, L_j \ge 0 && \forall e \in I, j \in J, t \in J.
\end{aligned} \tag{A.2}$$

The robust counterpart of constraints (7e) is

$$\begin{aligned}
&\sum_{e\in I} A_{ije}\bar{h}_e + D_{ij} - \sum_{e\in I} N_{ije} - \Gamma_h O_{ij} - \sum_{t\in J} P_{ijt} - kQ_{ij} \ge 0 && \forall i \in I, j \in J,\\
&- N_{ije} - O_{ij} \le A_{ije} h_e^{\Delta} && \forall i \in I, j \in J, e \in I,\\
&- P_{ijt} - Q_{ij} \le B_{ijt} && \forall i \in I, j \in J, t \in J,\\
&N_{ije}, O_{ij}, P_{ijt}, Q_{ij} \ge 0 && \forall i \in I, j \in J, e \in I.
\end{aligned} \tag{A.3}$$

The robust counterpart of constraints (7f) is

$$\begin{aligned}
&\sum_{e\in I} E_{ie}\bar{h}_e + G_i - \sum_{e\in I} R_{ie} - \Gamma_h K_i - \sum_{t\in J} T_{it} - kU_i \ge 0 && \forall i \in I,\\
&- R_{ie} - K_i \le E_{ie} h_e^{\Delta} && \forall i \in I, e \in I,\\
&- T_{it} - U_i \le F_{it} && \forall i \in I, t \in J,\\
&R_{ie}, K_i, T_{it}, U_i \ge 0 && \forall i \in I, e \in I, t \in J.
\end{aligned} \tag{A.4}$$

Therefore, the AARC model can be reformulated as

$$\begin{aligned}
\min_{\substack{\boldsymbol{y},\boldsymbol{\eta},\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\nu},\boldsymbol{\rho},\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{E},\boldsymbol{F},\boldsymbol{G}\\ \boldsymbol{S},\boldsymbol{H},\boldsymbol{L},\boldsymbol{N},\boldsymbol{O},\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R},\boldsymbol{K},\boldsymbol{T},\boldsymbol{U}}} \ \max_{(\boldsymbol{h},\boldsymbol{z})\in\mathcal{W}} \ & \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_{ij}(\sum_{e\in I} A_{ije} h_e + \sum_{t\in J} B_{ijt} z_t + D_{ij})\\
& + \sum_{i\in I} p_i(\sum_{e\in I} E_{ie} h_e + \sum_{t\in J} F_{it} z_t + G_i),\\
\text{s.t. } & y_j \in \{0,1\} \qquad \forall j \in J,\\
& \text{and (A.1)–(A.4).}
\end{aligned}$$

Since $\mathcal{W}$ is compact and convex, we can apply Sion's minimax theorem to reverse the order of minimization over $\{\boldsymbol{\eta},\boldsymbol{\mu},\boldsymbol{\sigma},\boldsymbol{\nu},\boldsymbol{\rho},\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{E},\boldsymbol{F},\boldsymbol{G},\boldsymbol{S},\boldsymbol{H},\boldsymbol{L},\boldsymbol{N},\boldsymbol{O},\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R},\boldsymbol{K},\boldsymbol{T},\boldsymbol{U}\}$ with the maximization over $\{\boldsymbol{h},\boldsymbol{z}\}$, and then replace the inner minimization by its dual maximization problem. The dual maximization problem and the maximization with respect to $\{\boldsymbol{h},\boldsymbol{z}\}$ can be merged to the following problem

$$\begin{aligned}
\max_{\substack{\boldsymbol{W},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{A}',\boldsymbol{B}',\boldsymbol{E}',\\ \boldsymbol{F}',\boldsymbol{G}',\boldsymbol{S}',\boldsymbol{H}',\boldsymbol{L}',\boldsymbol{N}'}} \ & \sum_{j\in J} f_j y_j + \sum_{i\in I}\bar{h}_i W_i + \sum_{i\in I}\sum_{\substack{e\in I,\\ e=i}} h_i^{\Delta} Y_{ie} - \sum_{j\in J} C_j y_j A'_j + \sum_{t\in J}\sum_{\substack{j\in J,\\ j=t}} C_j y_j E'_{tj}\\
\text{s.t. } & \bar{h}_e(W_i - A'_j + F'_{ij} - d_{ij}) + h_e^{\Delta}(Y_{ie} - B'_{ej} + G'_{ije} - d_{ij}\theta_e) = 0 && \forall i \in I, j \in J, e \in I,\\
& Z_{it} - E'_{tj} + S'_{ijt} = d_{ij} z_t && \forall i \in I, j \in J, t \in J,\\
& W_i - A'_j + F'_{ij} = d_{ij} && \forall i \in I, j \in J,\\
& \bar{h}_e(W_i + H'_i - p_i) + h_e^{\Delta}(Y_{ie} + L'_{ie} - p_i\theta_e) = 0 && \forall i \in I, e \in I,
\end{aligned}$$

$$
\begin{aligned}
& Z_{it} + N'_{it} = p_i z_t && \forall i \in I, t \in J,\\
& W_i + H'_i = p_i && \forall i \in I,\\
& -W_i + Y_{ie} \leq 0 && \forall i \in I, e \in I,\\
& -\Gamma_h W_i + Y_{ie} \leq 0 && \forall i \in I, e \in I,\\
& -W_i - Z_{it} \leq 0 && \forall i \in I, t \in J,\\
& -kW_i + \sum_{t \in J} Z_{it} \leq 0 && \forall i \in I,\\
& -A'_j + B'_{ej} \leq 0 && \forall e \in I, j \in J,\\
& -\Gamma_h A'_j + \sum_{e \in I} B'_{ej} \leq 0 && \forall j \in J,\\
& -A'_j + E'_{tj} \leq 0 && \forall t \in J, j \in J,\\
& -kA'_j + \sum_{t \in J} E'_{tj} \leq 0 && \forall j \in J,\\
& -F'_{ij} + G'_{ije} \leq 0 && \forall i \in I, j \in J, e \in I,\\
& -\Gamma_h F'_{ij} + \sum_{e \in I} G'_{ije} \leq 0 && \forall i \in I, j \in J,\\
& -F'_{ij} + S'_{ijt} \leq 0 && \forall i \in I, j \in J, t \in J,\\
& -kF'_{ij} + \sum_{t \in J} S'_{ijt} \leq 0 && \forall i \in I, j \in J,\\
& -H'_i + L'_{ie} \leq 0 && \forall i \in I, e \in I,\\
& -\Gamma_h H'_i + \sum_{e \in I} L'_{ie} \leq 0 && \forall i \in I,\\
& -H_i + N'_{it} \leq 0 && \forall i \in I, t \in J,\\
& -kH'_i + \sum_{t \in J} N'_{it} \leq 0 && \forall i \in I,\\
& 0 \leq \theta_i \leq 1 && \forall i \in I,\\
& 0 \leq z_j \leq 1 && \forall j \in J,\\
& \sum_{i \in I} \theta_i \leq \Gamma_h, &&\\
& \sum_{j \in J} z_j \leq k, &&\\
& W_i, Y_{ie}, Z_{it}, A'_j, B'_{ej}, E'_{tj}, F'_{ij}, G'_{ije}, S'_{ijt}, H'_i, L'_{ie}, N'_{it} \geq 0 && \forall i \in I, e \in I, j \in J, t \in J.
\end{aligned}
$$

We can rewrite the first constraints as

$$h_e^{\Delta}(Y_{ie} - B'_{ej} + G'_{ije} - d_{ij}\theta_e) = 0 \quad \forall i \in I, j \in J, e \in I,$$

because $W_i - A'_j + F'_{ij} = d_{ij}, \forall i \in I, j \in J$. Similarly, we can rewrite the fourth constraints as

$$h_e^{\Delta}(Y_{ie} + L'_{ie} - p_i\theta_e) = 0 \qquad \forall i \in I, e \in I,$$

because $W_i + H'_i = p_i, \forall i \in I$. We eliminate the constraints $-W_i - Z_{it} \leq 0, \forall i \in I, t \in J$ because they always hold as $W_i \geq 0, Z_{it} \geq 0$.

Appendix B Robust counterpart of the static RO model

The static RO model for the CFLP with facility disruptions is

$$
\begin{aligned}
\min_{\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}} \ \sup_{\boldsymbol{z}\in\mathcal{Z}_k} \ & \sum_{j\in J} f_j y_j + \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij} + \sum_{i\in I} p_i u_i, \\
\text{s.t.} \ & \sum_{j\in J} x_{ij} + u_i \geq \bar{h}_i && \forall i \in I, \\
& \sum_{i\in I} x_{ij} \leq C_j y_j (1 - z_j) && \forall \boldsymbol{z} \in \mathcal{Z}_k, j \in J, \\
& y_j \in \{0,1\} && \forall j \in J, \\
& x_{ij} \geq 0 && \forall i \in I, j \in J, \\
& u_i \geq 0 && \forall i \in I.
\end{aligned}
$$

Through duality theory, the second constraints can be reformulated as

$$
\begin{aligned}
& \sum_{i\in I} x_{ij} \leq C_j y_j - kA_j - B_j && \forall j \in J, && \text{(B.1)} \\
& C_j y_j - A_j - B_j \leq 0 && \forall j \in J, && \text{(B.2)} \\
& A_j, B_j \geq 0 && \forall j \in J. && \text{(B.3)}
\end{aligned}
$$

Constraints (B.2)–(B.3) indicate that when $k \geq 1$, the equation $C_j y_j - kA_j - B_j \leq 0$ always holds. Therefore, when $k \geq 1$, we have $\sum_{i\in I} x_{ij} \leq 0, \forall j \in J$, which suggests that $x_{ij} = 0, \forall i \in I, j \in J$ and $u_i = \bar{h}_i, \forall i \in I$. Since the static RO model is a minimization problem, all y_j would be 0 at optimality.

Appendix C Optimality of the affine policy for the CFLP-DR

We use duality theory to derive the subproblem of the C&CG algorithm, which is

$$
\begin{aligned}
\max & \sum_{j\in J} f_j \hat{y}_j + \sum_{i\in I} \alpha_i(\bar{h}_i + \theta_i h_i^{\Delta}) - \sum_{j\in J} C_j \hat{y}_j (1 - z_j)\beta_j \\
\text{s.t.} \ & \alpha_i - \beta_j \leq d_{ij} && \forall i \in I, j \in J, \\
& \alpha_i \leq p_i && \forall i \in I, \\
& \alpha_i, \beta_j \geq 0 && \forall i \in I, j \in J, \\
& \sum_{j\in J} z_j \leq k, \\
& \sum_{i\in I} \theta_i \leq \Gamma_h, \\
& z_j \in \{0,1\} && \forall j \in J, \\
& 0 \leq \theta_i \leq 1 && \forall i \in I.
\end{aligned}
\tag{C.1}
$$

Note that the second term in the objective is nonlinear (product of two continuous variables) and cannot be linearized. Thus, in Section 4.1, the subproblem is derived using the Karush–Kuhn–Tucker conditions. Here, we use the duality theory because it is easy to explain the proof process by using model (C.1).

When $\Gamma_h = |I|$, $\theta_i, \forall i \in I$ will take the value of 1, because it is a maximization problem and $\alpha_i \geq 0, \forall i \in I$. The resulting sub and master problems are the same as those of the CFLP-R. Thus, the CFLP-DR reduces to the CFLP-R with each customer's demand reaching the maximal value when $\Gamma_h = |I|$. Since the affine policy is optimal for the CFLP-R with $k = 1$, it is also optimal for the CFLP-DR with $k = 1$ and $\Gamma_h = |I|$.

When $k = |J|$, we can set $z_j = 1, \forall j \in J$ in each iteration of the C&CG algorithm to maximize the objective of the subproblem. Thus, in the third constraint of the master problem, $z_j^l, \forall j \in J, l \in \{1, \ldots, n\}$ will be 1. As the master problem is a minimization problem, we get $y_j = 0, \forall j \in J$ at optimality for the ARC model. In the AARC model, for any $i \in I$ and $j \in J$, we can let $A_{ije} = 0, B_{ijt} = 0, D_{ij} = 0$ to construct a corresponding solution $y_j = 0, x_{ij} = 0$ (u_i can adaptive to each scenario l). To conclude, when $k = |J|$, no matter the budget of demand uncertainty, both the exact method and the affine policy identify solutions with no opened facilities for the CFLP-DR.

Appendix D The two-stage stochastic programming model

Notation. o_s is the occurrence probability of scenario $s \in S$. x_{ij}^s and u_i^s are the corresponding decision variables in scenario s. h_i^s is the demand realization of customer i in scenario s. z_j^s is the status of facility j in scenario s.

The two-stage stochastic programming model is

$$
\begin{aligned}
\min_{\boldsymbol{y}, x^s, u^s} \quad & \sum_{j \in J} f_j y_j + \sum_{s \in S} o_s \Big(\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}^s + \sum_{i \in I} p_i u_i^s\Big), \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij}^s + u_i^s \geq h_i^s && \forall s \in S, i \in I, \\
& \sum_{i \in I} x_{ij}^s \leq C_j y_j (1 - z_j^s) && \forall s \in S, j \in J, \\
& y_j \in \{0, 1\} && \forall j \in J, \\
& x_{ij}^s \geq 0 && \forall s \in S, i \in I, j \in J, \\
& u_i^s \geq 0 && \forall s \in S, i \in I,
\end{aligned}
$$

where the objective minimizes the sum of facility construction cost and the expected transportation and penalty costs.