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On Uniqueness and Proportionality in Multi-Class Equilibrium Assignment

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Abstract. Over the past few years, much attention has been paid to computing flows for multi-class network equilibrium models that exhibit uniqueness of the class flows and proportionality (Bar-Gera et al, 2012). Several new algorithms have been developed such as origin based method (Bar Gera, 2002), bush based method (Dial, 2006), and LUCE (Gentile, 2012), that are able to obtain very fine solutions of network equilibrium models. These solutions can be post processed (Bar Gera, 2006) in order to ensure proportionality and class uniqueness of the flows. Recently developed, the TAPAS algorithm (Bar Gera 2010) is able to produce solutions that have proportionality embedded, without a post processing. It was generally accepted that these methods for solving UE traffic assignment are the only way to obtain unique path and class link flows. The purpose of this paper is to show that the linear approximation method and its bi-conjugate variant satisfy these conditions as well. In particular, some analytical results regarding the behaviour of the path flows entropy are presented which may be useful in an eventual theoretical proof that the linear approximation equilibrium flows maximize the entropy of the path flows.

Keywords. Network equilibrium, path flows entropy, uniqueness of path flows, proportionality.

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1. Introduction

The network route choice models most commonly used in transportation planning methods are steady state models, in spite of the fact that all traffic phenomena are temporal. A given period of time for which the demand for travel is quantified is considered and then the flow pattern which results from the action of the demand and the performance of the transport infrastructure available needs to be determined. Deterministic network equilibrium assignment model of route choice are usually based on Wardrop's (1952) user optimal principle. Based on the seminal work of Beckmann (1956) a large body of research and literature on the structure and solution algorithms for various versions of the network equilibrium model (variable demand, multi-class, asymmetric cost functions) has been contributed (see for instance Patriksson, 1994, Florian and Hearn, 1995, Marcotte and Patriksson, 2007).

Over the past few years, much attention has been paid to computing flows for multi-class network static equilibrium models that exhibit proportionality and hence uniqueness of the path (also known as route) and class flows (Bar-Gera et. al, (2012)). Several fast converging algorithms have been developed, as alternatives to the slow converging adaptation of the linear approximation method (Frank and Wolfe, 1956) for computing network equilibrium flows. These include the *origin based* method (Bar Gera, 2002) and the *bush based* methods (Dial 2006), and LUCE (Gentile, 2012), that are able to obtain very fine solutions of network equilibrium models. These solutions can be post-processed (see Bar Gera, 2006) in order to ensure proportionality and class uniqueness of the flows. The proportionality property ensures that all the path flows, from all O-D pairs, when splitting between the same two alternative route segments (sub-paths), will be distributed over two alternative route segments in the same proportions as the demand. This assumption (Bar-Gera and Boyce (1999)) is a sufficient condition that characterizes entropy maximizing path flows. A more recent development, the TAPAS algorithm (Bar Gera, 2010) is able to produce solutions that have proportionality embedded, without requiring post processing. It was generally accepted (Boyce and Xie, 2013) that these methods for solving for user equilibrium flows are the only way to obtain unique path and class link flows. A comparative study of TAPAS and several commercially available versions of the linear approximation for solving network equilibrium problems (Boyce et al. (2010) or Bar Gera et al (2012)) present comparisons between them in terms of proportionality. These results indicate that the linear approximation method results are close to proportionality, but not close enough. It is worth mentioning that in this study, a single class assignment was carried out on the Chicago network to a relative gap of 10^{-4} with the linear approximation method and from $\sim 10^{-4}$ to $\sim 10^{-12}$ with TAPAS. Recently, the convergence of the linear approximation algorithm (Frank-Wolfe (1956)) has been improved with the conjugate / bi-conjugate variant of Mitradjieva and Lindberg (2013). This allows one to solve the traffic equilibrium problem to a finer solution, which in turn significantly improves the proportionality and class flows uniqueness of the obtained solution.

In this paper, we show, by comparing the results of a two-class assignment obtained with TAPAS to that of the linear approximation and the bi-conjugate variant of the linear approximation method, that these methods exhibit, within the solution precision, both class uniqueness and proportionality. This discovery is very useful since the bi-conjugate variant of the linear approximation method can be multi-threaded and executed on multiprocessor computing platforms, requiring small data storage. Therefore, it provides a more attractive and computationally efficient method for solving multi-class assignments that bush based methods, for convergence levels used in practice.

In the next section some numerical results are presented, which provide empirical evidence of near proportionality of the equilibrium flows and unique class flows for a two-class instance. Section 3 presents some theoretical results regarding the path flows entropy value during the linear approximation algorithm. The paper ends with a short conclusion and acknowledgments.

2. Some empirical evidence of proportionality

The computational experiments reported here are using the Chicago test database with two classes of traffic, cars and trucks, that was used in the application of a bush based method by Boyce and Xie (2013). This data as well class flows obtained by the execution of the TAPAS code (Bar Gera (2010)) to a convergence criterion of relative gap of less than 10^{-12} were kindly made available to us by David Boyce.

This database is widely used as a benchmark for the traffic assignment algorithms (see Figure 1). It has 1,790 zones 11,192 nodes and 39,018 links. There are 563 links where trucks are not permitted.

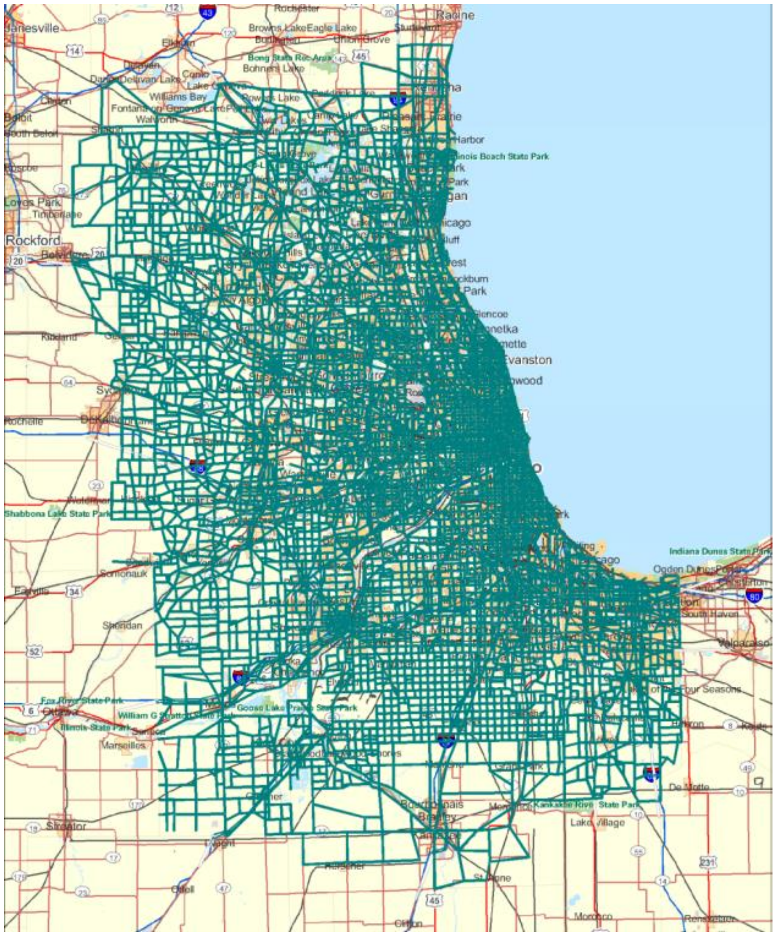

Figure 1. The Chicago test network

We ran the same two-class assignment by using both a multi-threaded version of the linear approximation method to a relative gap of $2.3\cdot10^{-6}$ and a multithreaded variant of the bi-conjugate linear approximation method[1] up to a relative gap of 10^{-6}. Figures 2 and 3 show plots of car and truck link flows obtained with a linear approximation method versus the flow obtained with TAPAS. Figures 4 and 5 show plots of car and truck link flows obtained with the bi-conjugate variant of the linear approximation method versus the flow obtained with TAPAS. The computation of this two-class assignment required approximately 3.5 hours with the linear approximation method and 18 minutes with the bi-conjugate variant on a hyper-threaded 16 Xeon, 2.9 Ghz processor computing platform, using 32 threads.

[1] Implemented as SOLA in Emme 4.1 software package.

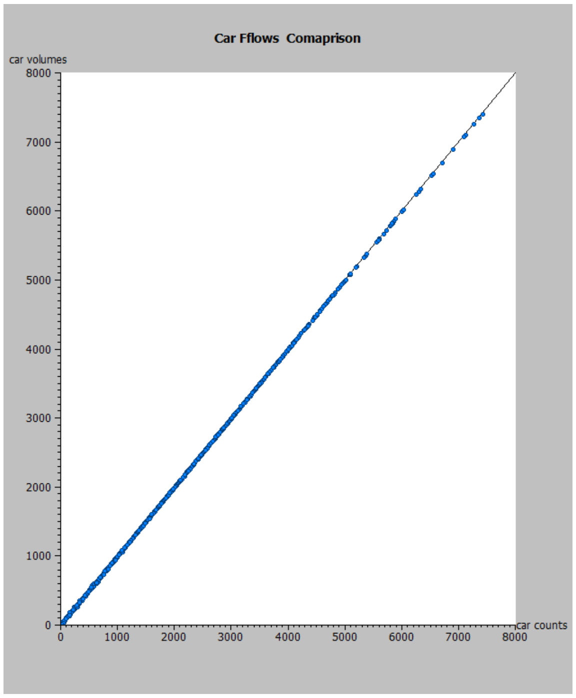

Figure 2. Comparison of car flows (linear approximation vs. TAPAS)

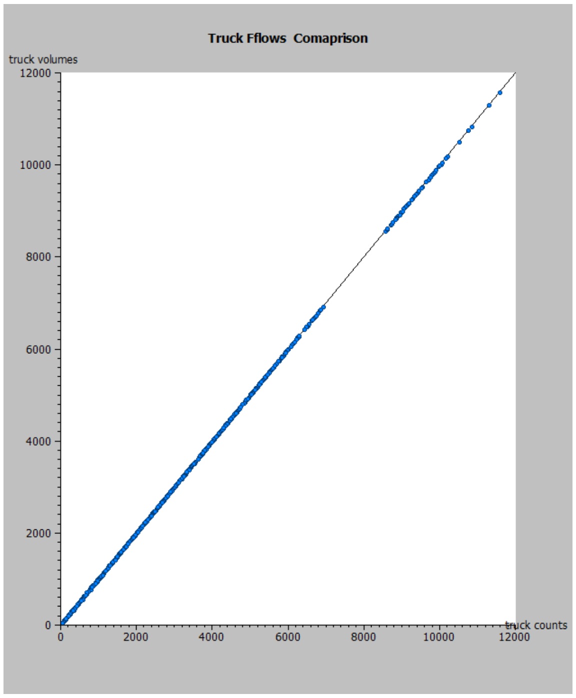

Figure 3. Comparison of truck flows (linear approximation vs. TAPAS)

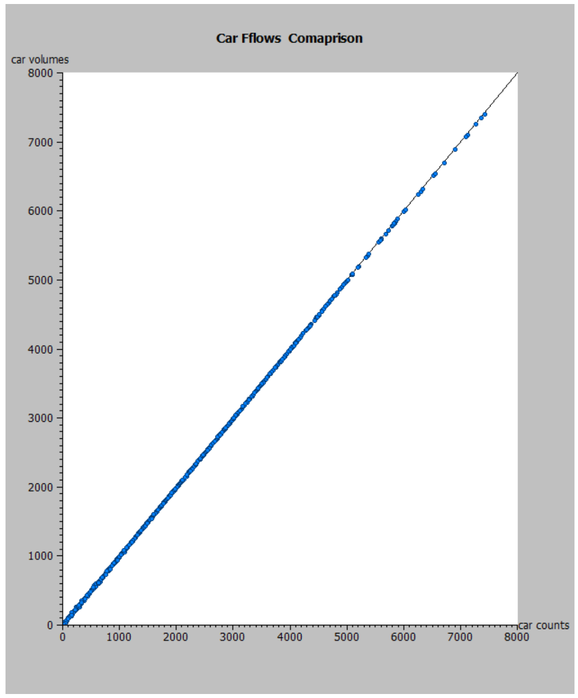

Figure 4. Comparison of car flows (bi-conjugate linear approximation vs. TAPAS)

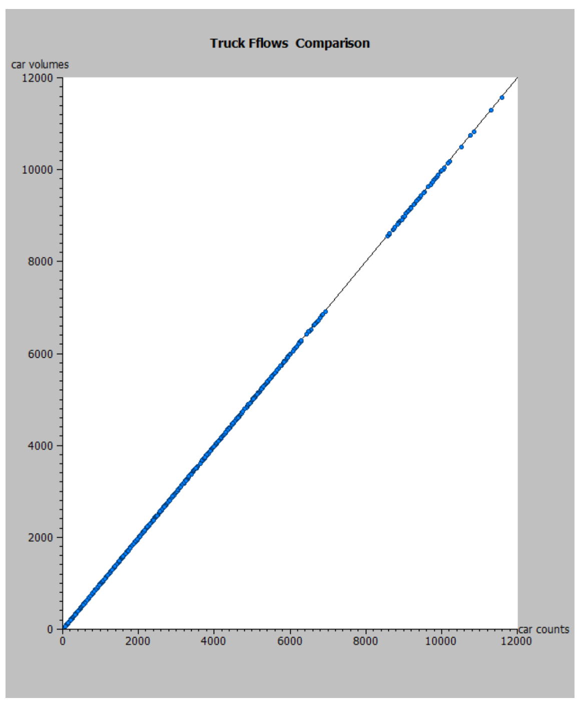

Figure 5. Comparison of truck flows (bi-conjugate linear approximation vs. TAPAS)

Figures 2, 3, 4, and 5 clearly show that both the linear approximation method and the bi-conjugate variant are producing almost the same class flows as TAPAS. Since the assignments are not run to the same precision, the fit is not perfect. The bi-conjugate variant of the linear approximation converges in a reasonable time to a relative gap of 10^{-6} and it was not considered necessary to obtain a relative gap of 10^{-7} in view of these results.

We also studied the proportionality property using the two methods of solving for equilibrium. Recall that this property assumes that all the path flows, from all O-D pairs, when splitting between the same two alternative route segments (sub-paths), it will be distributed over two alternative route segments in the same proportions as the demand.. To help verify this property, the ratio of travelers traversing the lower to upper alternative route segment, should form a straight line on the chart that plots O-D demand that uses each segment.

For that purpose we analyzed a pair of alternative segments identified by Bar-Gera, H., Boyce, D. and Nie, Y., in their study report of 2012. The flow on the pair of alternative segments is shown in Figure 6. The O-D pairs that contribute flows to each segment were computed by appropriate path analyses.

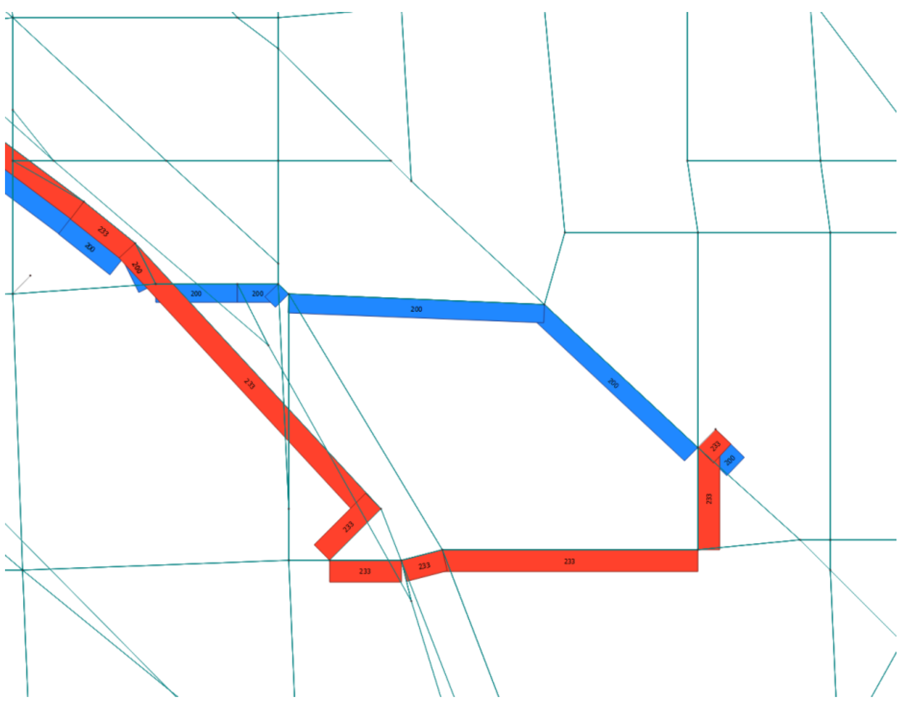

Figure 6. Flows on a pair of alternative segments

The charts from Figure 7 and 8 show plots of the O-D demands that contribute to the flow of each one of the segments for cars and trucks. The relatively straight line of these plots indicates that the condition of proportionality is approximately satisfied. In Figure 7, the matrices of the O-D demand that contribute flows to the two segments of the pair alternative segments for car flows are plotted. In Figure 8, the matrices of the O-D demand that contribute flows to the two segments of the pair alternative segments for truck flows are plotted. The proportionality is not perfect but may be considered to be close enough.

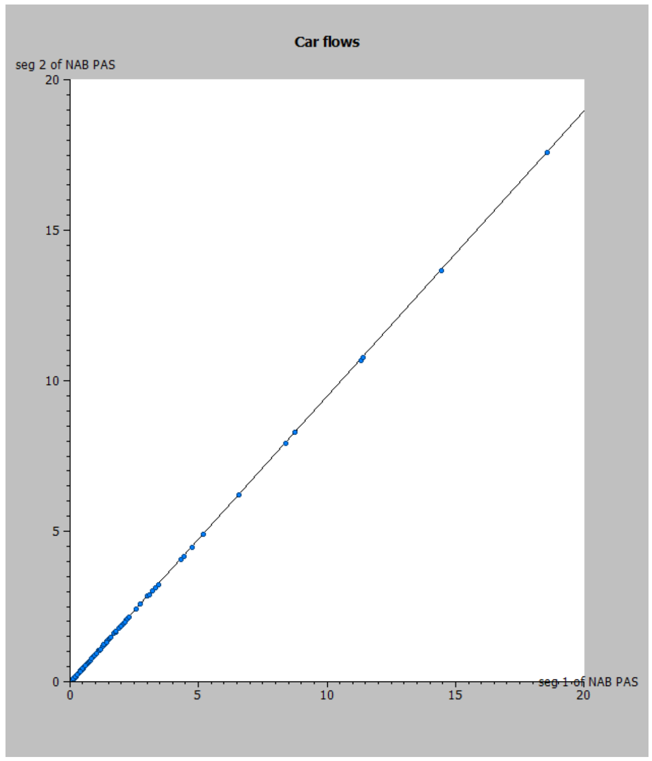

Figure 7. O-D pairs using each alternative segment for cars

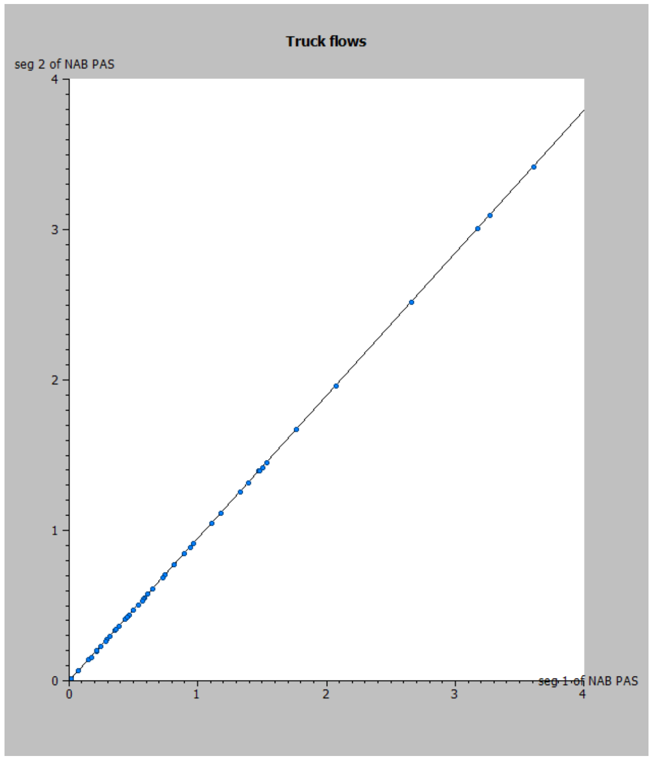

Figure 8. O-D pairs using each alternative segment for trucks

Two more O-D matrices, referred to as M2 and M3, were provided for the Chicago network, each resulting in an increasingly congested network. The TAPAS flows for an equilibrium assignment with a relative gap of 10^{-12} were also made available to us by David Boyce. The total vehicle hours of the equilibrium flows obtained with first O-D matrix are 330,815, while the use of O-D matrices M2 and matrices results in total vehicle-hours of 433,222 and 568,362 respectively. The more congested the network, the more iterations are required to obtain the equilibrium flows. The computation times for these two assignments were 32 and 51 minutes respectively in order to attain a relative gap of 10^{-6}. The comparison of the flows obtained for cars and trucks with the bi-conjugate variant versus the TAPAS flows are shown in Figures 9, 10, 11, and 12.

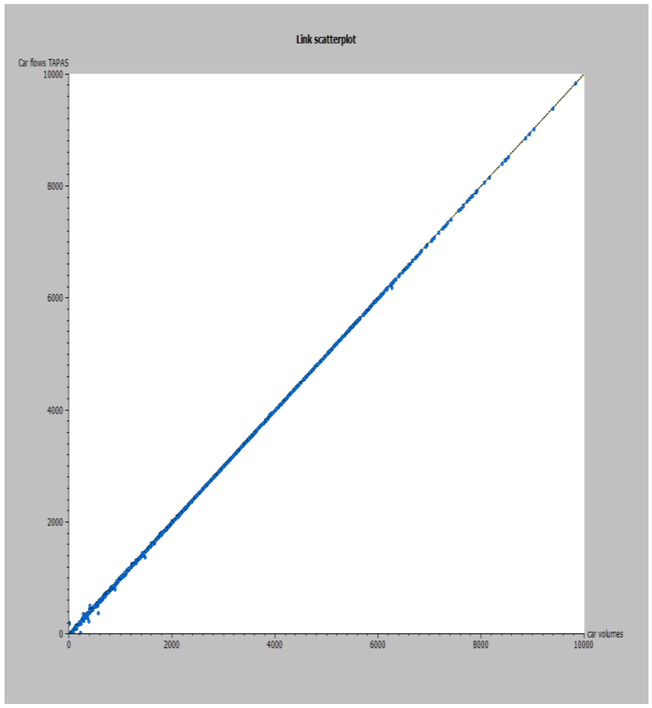

Figure 9. Comparison of car flows (bi-conjugate linear approximation vs. TAPAS) Matrix M2

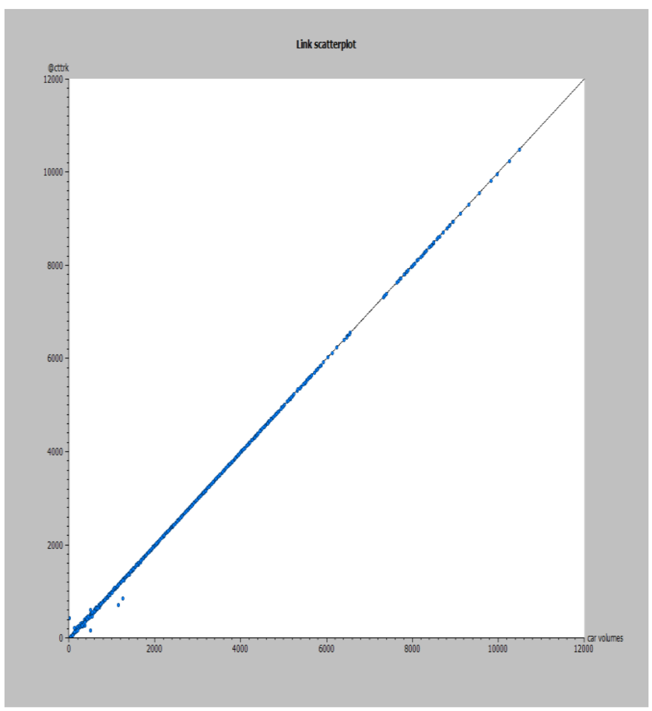

Figure 10. Comparison of truck flows (bi-conjugate linear approximation vs. TAPAS) Matrix M2

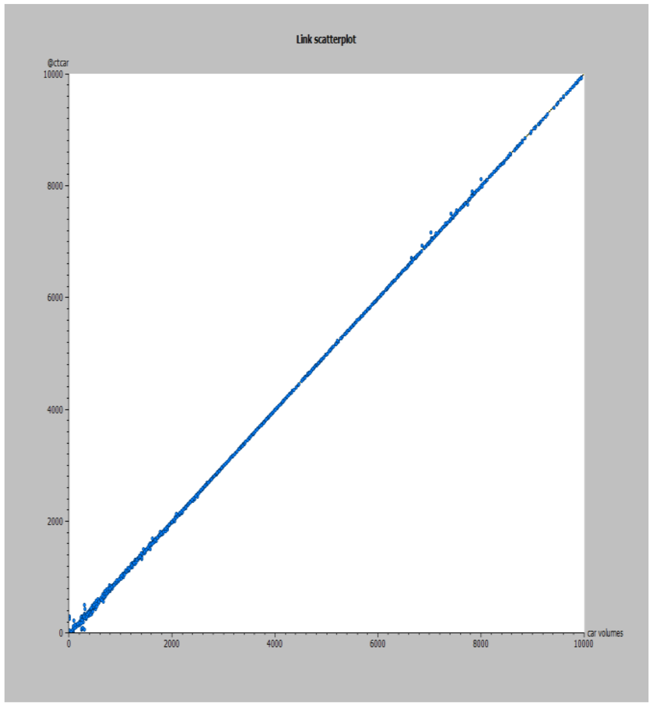

Figure 11. Comparison of car flows (bi-conjugate linear approximation vs. TAPAS) Matrix M3

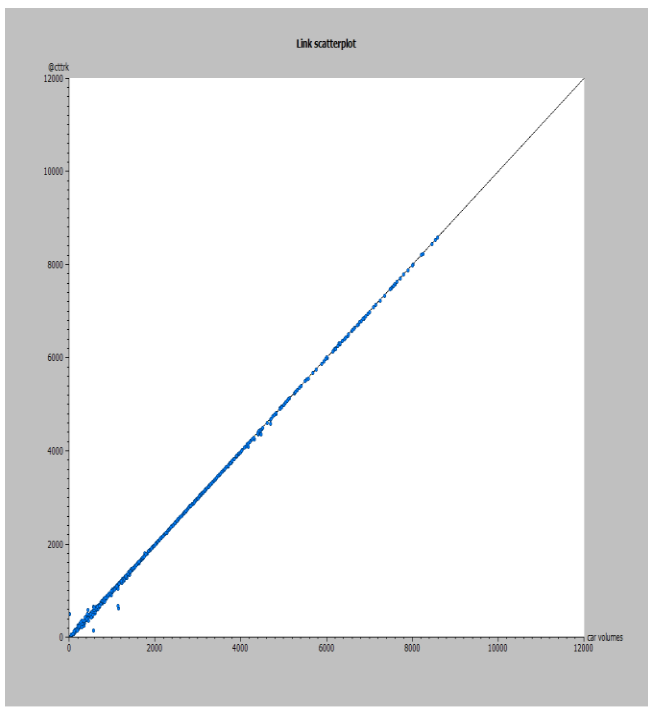

Figure 12. Comparison of truck flows (bi-conjugate linear approximation vs. TAPAS) Matrix M3

It is rather evident that the flows produced by the two algorithms are nearly the same: large valued flows are practically equal but some of the lower valued flows differ from the TAPAS flows. This is probably due to the higher level of congestion generated by these O-D matrices.

3. Some analytical results

In order to introduce the analytical results, some minimal notation is required. A transportation network is modelled as a directed, weighted graph $G=(N,A)$, which has origins $p \in P \subset N$ and destinations $q \in Q \subset N$. The links $a \in A$ carries positive flow v_a, which is used to establish the link weight via monotonically increasing positive cost functions $s_a\left(v_a\right)$. The origin to destination demands for each origin-destination (O-D) pair pq are g_{pq} and give rise to path flows h_k, on paths $k \in K_{p,q}$. The one-class, static traffic equilibrium problem can be formulated as a non-linear convex optimization program (Beckmann et al (1956)):

$$\begin{aligned}
&Min \sum_{a \in A} \int_0^{v_a} s_a(x)dx, \\
&\text{subject to:} \\
&\sum_{k \in K_{pq}} h_k = g_{pq}, \ \ p \in P, \ q \in Q \\
&v_a = \sum_{a \in A} \delta_{ak} h_k, \ a \in A \\
&v_a \geq 0, \ a \in A
\end{aligned} \tag{1}$$

For non-trivial instances of the problem, the link equilibrium flows vector is unique but the path flows are not necessarily unique. This can be easily verified by using the Karush-Kuhn-Tucker conditions. Nevertheless, maximizing the path flows entropy makes the solution in the paths space unique (see Lu and Nie (2010), for example).

Among the many algorithms developed to solve the above optimization problem, the first one, and the most commonly used over the years is the linear approximation method of Frank and Wolf (1956). The adaptation of this algorithm for solving the network equilibrium model and its bi-conjugate variant were used in the empirical tests reported above. It has the advantages of modest data storage requirements and suitability for parallelization. The generic adaptation of the linear approximation method to solve (1) is relevant for the following and is stated as follows:

Step 0. *Initialization*

An initial solution $\mathbf{v}^0$ is obtained by an all-or-nothing assignment of the demand $\mathbf{g}$ on shortest paths computed with arc costs $\mathbf{s}^0 = \mathbf{s}\left(0\right)$. Set iteration $k=0$.

Step 1. *Update link costs*

$$k = k+1;\ \mathbf{s}^k = \mathbf{s}\left(\mathbf{v}^{k-1}\right).$$

Step 2. *Compute descent direction*

Find extreme solution $\mathbf{y}^k$, which is the all-or-nothing assignment of demand $\mathbf{g}$ on the shortest paths, computed with arc costs $\mathbf{s}^k$; compute the descent direction $\mathbf{d}^k = \mathbf{y}^k - \mathbf{v}^{k-1}$.

Step 3. *Compute optimal step size*

Compute the optimal step $\lambda^k \in (0,1]$ on the line starting at $\mathbf{v}^{k-1}$ in direction $\mathbf{d}^k$.

Step 4. *Update link flows*

$$\mathbf{v}^k = \mathbf{v}^{k-1} + \lambda \cdot \mathbf{d}^k = (1-\lambda)\cdot \mathbf{v}^{k-1} + \lambda \cdot \mathbf{y}^k$$

Step 5. *Stopping criterion*

If a stopping criterion is satisfied, STOP; otherwise return to Step 1.

An important contribution by Bar-Gera and Boyce (1999) was to link the entropy measure of the path flows

$$E(\mathbf{h}) = -\sum_{pq}\sum_{k\in K_{p,q}} h_k\left(\ln\frac{h_k}{g_{pq}} - 1\right) \tag{2}$$

to the condition of proportionality. Essentially, it is proved that, for a given solution of the network equilibrium link flows, maximizing the path flows entropy implies the proportionality property.

Even though the results presented in Section 2 strongly suggest that the path flow entropy is maximized due to the uniqueness of the class flows and the proportionality obtained by using a linear approximation method, the entropy is not monotonically increasing at each iteration of the linear approximation method. Consider the three-link network in Figure 13, which contains one O-D pair with a demand of 1000 trips from *p* to *q*.

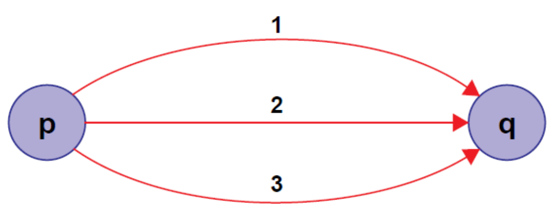

Figure 13. Three-link network

The link cost functions are:

$$s_1(v_1) = 10\left(1 + 0.15\left(v_1/200\right)^4\right)$$
$$s_2(v_2) = 20\left(1 + 0.15\left(v_1/400\right)^4\right) \quad (3)$$
$$s_1(v_1) = 25\left(1 + 0.15\left(v_1/300\right)^4\right)$$

The link flows and the path flows entropy obtained after the first nine iterations of the linear approximation method are shown in Table 1; the last row corresponds to the optimal solution, where all paths used are of equal cost. In this simple example, the entropy converges to the value corresponding to its optimal solution, but it is not monotonically increasing (see the highlighted cells).

Table 1. Link flows and path flows entropy at each iteration

Iteration k	**v_1**	**v_2**	**v_3**	**step size**	**entropy**
0	1000	0	0	1.00000	1000.000000
1	403	597	0	0.59654	1674.209328
2	338	500	161	0.16113	2006.247859
3	362	483	155	0.03555	2008.301137
4	355	473	173	0.02040	2026.289654
5	359	469	171	0.00719	2023.877620
6	357	467	176	0.00536	2029.062658
7	359	466	175	0.00200	2028.614491
8	358	465	177	0.00156	2030.298574
9	358	465	177	0.00059	2030.298574
...	...	...	...	...	...
optimal	**358**	**465**	**177**	**-**	**2030.298574**

In the following, some properties of the path flows entropy during a linear approximation type algorithm are derived.

***Proposition* 1**. *During a linear approximation algorithm, as long as only new shortest paths are discovered for a given O-D pair, the corresponding entropy at iteration n is*:

$$E_{pq}^{n} = -g_{pq}\sum_{i=0}^{n}\alpha_i\left(\ln\alpha_i - 1\right), \quad (4)$$

where

$$\alpha_i = \lambda_i \prod_{j=i+1}^{n} \left(1-\lambda_j\right), \tag{5}$$

g_{pq} *is the demand,* λ_i *is the step size at iteration* i, $\lambda_0 = 1$.

Proof

At iteration 0, the step size is 1, that is, the entire demand is assigned to one path to obtain an all-or-nothing assignment. This path flow is $h_0 = \lambda_0 \cdot g_{pq} = g_{pq}$. The corresponding entropy is:

$$E_{pq}^{0} = -h_0 \left(\ln \frac{h_0}{g_{pq}} - 1 \right) = h_0 \tag{6}$$

At iteration 1, assuming that a new path has been discovered, and a step size λ_1 has been computed, the flow of path 0 is weighted by $\left(1-\lambda_1\right)$ and a the new path is added with a weight of λ_1. The demand will decompose into two paths with flows $h_o = \left(1-\lambda_1\right) \cdot g_{pq}$ and

$h_1 = \lambda_1 \cdot g_{pq}$. Their corresponding entropy will be

$$\begin{aligned} E_{pq}^{1} &= -g_{pq} \cdot \left(1-\lambda_1\right)\left(\ln\left(1-\lambda_1\right)-1\right) - g_{pq} \cdot \lambda_1 \left(\ln \lambda_1 - 1\right) = \\ &= -g_{pq} \cdot \left(\left(1-\lambda_1\right)\left(\ln\left(1-\lambda_1\right)-1\right) + \lambda_1 \left(\ln \lambda_1 - 1\right)\right) = \\ &= -g_{pq} \cdot \left(\alpha_0 \left(\ln \alpha_0 - 1\right) + \alpha_1 \left(\ln \alpha_1 - 1\right)\right), \end{aligned} \tag{7}$$

with $\alpha_0 = 1 - \lambda_1$ and $\alpha_1 = \lambda_1$.

Assume that at a iteration n, the path decomposition is $g_{pq} = \sum_{i=0}^{n} h_i$, with $h_i = \alpha_i \cdot g_{pq}$, and $\alpha_i = \lambda_{n-i} \prod_{j=i+1}^{n} \left(1-\lambda_j\right)$. If at iteration $n+1$ a new path is discovered, its weight will be λ_{n+1}, whereas all the previous paths will be weighted by $\left(1-\lambda_{n+1}\right)$. Denoting by α_i' the coefficients of the newly obtained paths, the path flows decomposition at iteration $n+1$ can be written as $g_{pq} = \sum_{i=0}^{n+1} \alpha_i' \cdot g_{pq}$, where

$$\alpha_i' = \alpha_i \cdot \left(1-\lambda_{n+1}\right) = \lambda_i \cdot \left(1-\lambda_{n+1}\right) \cdot \prod_{j=i+1}^{n} \left(1-\lambda_j\right) = \lambda_n \prod_{j=i+1}^{n+1} \left(1-\lambda_j\right) \tag{8}$$

for all $i \in \{0, n\}$, and $\alpha_{n+1}' = \lambda_{n+1}$, which concludes the proof. □

Proposition 1 mainly states that, **a)** the path flows entropy for a given OD pair is directly proportional to the demand of that OD pair, and **b)** as long as only new shortest paths are discovered during a linear approximation algorithm, the constant for the direct proportionality of the path flows entropy of a given OD pair is an algebraic combination of the step sizes, regardless of a particular network topology.

Proposition 1 does not give direct information on the variations of the path flows entropy value during an assignment. Moreover, if the same path is discovered again at a certain iteration, the path entropy for an O-D pair cannot be expressed as done in Proposition 1. This is due to the fact that splitting the flow on the same path into two will not split the corresponding entropy into the same proportion. Consider a path with flow h; its corresponding entropy is

$$E = -h\left(\ln\frac{h}{g_{pq}} - 1\right) \tag{9}$$

Splitting the path flow into two $h = h_1 + h_2$, the corresponding entropy becomes

$$\begin{aligned} E' &= -h_1\left(\ln\frac{h_1}{g_{pq}} - 1\right) - h_2\left(\ln\frac{h_2}{g_{pq}} - 1\right) = \\ &= E + h\cdot\ln\frac{h}{g_{pq}} - h_1\cdot\ln\frac{h_1}{g_{pq}} - h_2\cdot\ln\frac{h_2}{g_{pq}} > E\,. \end{aligned} \tag{10}$$

Therefore, Proposition 1 provides an overestimate of the entropy when the same path is discovered. Moreover, the path flows entropy might decrease, in such a case. The following two propositions identify the conditions for the path entropy to increase at a given iteration if a new shortest path or if the same shortest path is discovered.

***Proposition* 2**. *During a linear approximation algorithm, if a new path is discovered for a given O-D pair at iteration n, the path entropy increases if the step size* λ *satisfies*

$$\lambda\sum_{i=1}^{m}\alpha_i\ln\alpha_i > \lambda\ln\lambda + (1-\lambda)\ln(1-\lambda), \tag{11}$$

where α_i *are the path flows proportions at iteration* $n-1$, $\sum_{i=1}^{m}\alpha_i = 1,\ 0\le m\le n$.

Proof

Assume that, at iteration $n-1$, the demand is decomposed into *m* paths, $g_{pq} = \sum_{i=1}^{m} h_i$, with $h_i = g_{pq}\cdot\alpha_i$, $m\le n$. The corresponding path flows entropy is

$$E_{pq}^{n-1} = -\sum_{i=1}^{m} h_i \left(\ln \frac{h_i}{g_{pq}} - 1 \right) = -g_{pq} \sum_{i=1}^{m} \alpha_i \left(\ln \alpha_i - 1 \right). \tag{12}$$

Applying a step size of λ on a new path, the path flows proportions become $\alpha_i' = \alpha_i (1 - \lambda)$ $i \in \{1, m\}$, and $\alpha_{m+1}' = \lambda$. The corresponding entropy will be

$$E_{pq}^{n} = -\sum_{i=1}^{m+1} h_i' \left(\ln \frac{h_i'}{g_{pq}} - 1 \right) =$$

$$= -g_{pq} \left(\lambda (\ln \lambda - 1) + \sum_{i=1}^{m} \alpha_i (1 - \lambda) \left(\ln \left(\alpha_i (1 - \lambda) \right) - 1 \right) \right). \tag{13}$$

Imposing $E_{pq}^{n} > E_{pq}^{n-1}$ implies

$$-g_{pq} \left(\lambda (\ln \lambda - 1) + \sum_{i=1}^{m} \alpha_i (1 - \lambda) \left(\ln \left(\alpha_i (1 - \lambda) \right) - 1 \right) \right) > -g_{pq} \sum_{i=1}^{m} \alpha_i (\ln \alpha_i - 1), \tag{14}$$

or equivalent

$$\sum_{i=1}^{m} \alpha_i (\ln \alpha_i - 1) - \lambda (\ln \lambda - 1) - \sum_{i=1}^{m} \alpha_i (1 - \lambda) \left(\ln \left(\alpha_i (1 - \lambda) \right) - 1 \right) > 0. \tag{15}$$

Taking into account that $\sum_{i=1}^{m} \alpha_i = 1$ and separating the terms in λ the claimed inequality (11) is obtained.□

***Proposition* 3**. *During a linear approximation algorithm, if path $m \le n$ is discovered again for a given O-D pair at iteration n, the corresponding path entropy increases if the step size λ satisfies*

$$\lambda \sum_{i=1}^{m} \alpha_i \ln \alpha_i > (\alpha_m + \lambda \beta_m) \ln (\alpha_m + \lambda \beta_m) - (1 - \lambda) \left(\alpha_m \ln \alpha_m - \beta_m \ln (1 - \lambda) \right), \tag{16}$$

where α_i are the path flows proportions at iteration $n-1$, $\beta_m = 1 - \alpha_m$, $\sum_{i=1}^{m} \alpha_i = 1$, with $0 \le m \le n$.

Proof

Assume that, at iteration $n-1$, the demand is decomposed into *m* paths, $g_{pq} = \sum_{i=1}^{m} h_i$, with $h_i = g_{pq} \cdot \alpha_i$, $m \le n$. The corresponding path flows entropy is

$$E_{pq}^{n-1} = -\sum_{i=1}^{m} h_i \left(\ln \frac{h_i}{g_{pq}} - 1 \right) = -g_{pq} \sum_{i=1}^{m} \alpha_i \left(\ln \alpha_i - 1 \right). \tag{17}$$

Without loss of generality, assume that path m is discovered again. Applying a λ step size on this path, the path flows proportions become $\alpha_i' = \alpha_i(1-\lambda)$, for $i \in \{1, m-1\}$, and $\alpha_m' = (1-\lambda)\alpha_m + \lambda = \alpha_m + \lambda(1-\alpha_m) = \alpha_m + \lambda \cdot \beta_m$. The corresponding entropy will be

$$E_{pq}^n = -\sum_{i=1}^{m} h_i' \left(\ln \frac{h_i'}{g_{pq}} - 1 \right) =$$

$$= -g_{pq}\left(\left((\alpha_m + \lambda\beta_m)\left(\ln(\alpha_m + \lambda\beta_m) - 1 \right) \right) + \sum_{i=1}^{m-1} \alpha_i (1-\lambda)\left(\ln(\alpha_i(1-\lambda)) - 1 \right) \right). \tag{18}$$

Imposing $E_{pq}^n > E_{pq}^{n-1}$ implies

$$\left((\alpha_m + \lambda\beta_m)\left(\ln(\alpha_m + \lambda\beta_m) - 1 \right) \right) + \sum_{i=1}^{m-1} \alpha_i (1-\lambda)\left(\ln(\alpha_i(1-\lambda)) - 1 \right) < \sum_{i=1}^{m} \alpha_i (\ln \alpha_i - 1). \tag{19}$$

Taking into account that $\sum_{i=1}^{m} \alpha_i = 1$, denoting $\beta_m = 1-\alpha_m$ and rearranging the terms the claimed inequality (16) is obtained.□

Note that properties 2 and 3 also give some information about the variation of the entropy between two consecutive iterations close to the equilibrium. It is clear that the step size tends to 0 close to the equilibrium. Starting from inequalities (11) and (16), define the following functions of the step size:

$$f_2(\lambda) = \lambda \sum_{i=1}^{m} \alpha_i \ln \alpha_i - \lambda \ln \lambda - (1-\lambda)\ln(1-\lambda) \tag{20}$$

and

$$f_3(\lambda) = \lambda \sum_{i=1}^{m} \alpha_i \ln \alpha_i - (\alpha_m + \lambda\beta_m)\ln(\alpha_m + \lambda\beta_m) + (1-\lambda)\left(\alpha_m \ln \alpha_m - \beta_m \ln(1-\lambda) \right), \tag{21}$$

which cover both cases of a linear approximation iteration: when a new shortest path or a previously known path are discovered at a certain iteration. It can be seen that $\lim_{\lambda \to 0} f_2(\lambda) = 0$ and $\lim_{\lambda \to 0} f_3(\lambda) = 0$, that is, the variation in the path flows entropy approaches to 0 as the step size tends to 0.

The three analytical properties presented above may be useful in providing a rigorous proof that the equilibrium flows obtained with the linear approximation algorithm maximize the path flows entropy. They mainly show that the path flows entropy value of an OD pair during the linear approximation algorithm iterations is directly proportional only to the demand of that O-D pair. As this value can increase or decrease at certain iterations, a rigorous proof that the total path flows entropy converges to a maximum value, as the experiments presented in this paper suggest, is still an open problem.

4. Conclusions

From the empirical results provided in this paper, it can be safely concluded that linear approximation method and its conjugate variants yield approximately unique path and class flows since they approximately satisfy the condition of proportionality. This is a new finding as since it was not known that some of the linear approximation algorithms used in this study possess these properties. The computing times that can be realized on multi-processor computing platforms for convergence levels of up to 10^{-6} render the bi-conjugate variant of the linear approximation method an attractive alternative for solving large scale multi-class assignments problems on which bush based methods are still relatively untested. The flow comparisons between equilibrium flows at a relative gap of 10^{-6} and a relative gap of less than 10^{-12} shed some light on the benefit of computing equilibrium flows with very small relative gaps.

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