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# Dynamic Facility Location with Generalized Modular Capacities

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**Abstract.** Location decisions are frequently subject to dynamic aspects such as changes in customer demand. Often, flexibility regarding the geographic location of facilities, as well as their capacities, is the only solution to such issues. Even when demand can be forecast, finding the optimal schedule for the deployment and dynamic adjustment of capacities remains a challenge, especially when the cost structure for these adjustments is complex. In this paper, we introduce a unifying model that generalizes existing formulations for several dynamic facility location problems and provides stronger linear programming relaxations than the specialized formulations. In addition, the model can address facility location problems where the costs for capacity changes are based on a cost matrix. To the best of our knowledge, this problem has not been addressed in the literature. We apply our model to special cases of the problem with capacity expansion and reduction or temporary facility closing and reopening. We prove dominance relationships between our formulation and existing models for the special cases. Computational experiments on a large set of randomly generated instances with up to 100 facility locations and 1000 customers show that our model can obtain optimal solutions in shorter computing times than the existing specialized formulations.

**Keywords**. Mixed-integer programming (MIP), facility location, modular capacities.

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# 1 Introduction

Dynamic facility location consists in deciding *where* and *when* to provide capacity to satisfy customer demand at the lowest cost. This demand is rarely stable, but rather increases, decreases or oscillates over time. Therefore, facility capacities often have to be adjusted dynamically. Many variants of dynamic facility location problems have been studied, suggesting different ways to adjust capacities throughout a given planning horizon. The most common features include capacity expansion and reduction (Luss, 1982; Mirchandani and Francis, 1990; Peeters and Antunes, 2001; Troncoso and Garrido, 2005; Dias et al., 2007), temporary facility closing (Chardaire et al., 1996; Canel et al., 2001; Dias et al., 2006), as well as the relocation of capacities (Melo et al., 2005). Mathematical models that include such features have been applied in both the private and the public sectors to determine locations and capacities for production facilities, schools, hospitals, libraries and many more.

Many of these studies also acknowledged the existence of economies of scale (Correia and Captivo, 2003; Correia et al., 2010). While previous works considered economies of scale mainly for the construction and production costs, the costs for adjusting the capacities of the facilities have commonly been modeled in less detail. However, the latter is necessary to ensure a fair representation of the cost structure found in practice. The costs to adjust capacities often do not only depend on the size of the adjustment, but also on the current capacity level. This is true in a large class of applications, especially in transportation, logistics and telecommunications, where additional capacity gets cheaper (or more expensive) when approaching the maximum capacity limit.

In this work, we introduce a very general dynamic facility location problem, referred to as the *Dynamic Facility Location Problem with Generalized Modular Capacities (DFLPG)*. The problem allows modular capacity changes subject to a detailed cost structure and is modeled as a mixed-integer programming (MIP) formulation. Due to its generality, this model unifies several existing problems found in the literature. The cost structure used in the model is based on a matrix describing the costs for capacity changes between all pairs of capacity levels. We are not aware of any other work dealing with facility location with a similar level of detail in the cost structure.

Our study is motivated by an industrial project with a Canadian logging company that must locate camps to host workers involved in wood harvest activities while optimizing the overall logistics and transportation costs (Jena et al., 2012). In this problem, the total capacity of a camp is represented by its number of hosting units, while additional units provide supporting infrastructure. As the relation between the number of different units is non-linear, the costs for capacity changes are described in a transition matrix.

The contribution of this work is three-fold. First, we introduce a general dynamic facility location model that comprises a large set of existing formulations. Second, we analyze the linear programming (LP) relaxation bound obtained by our model, showing that it is at least as strong as the LP relaxation bound of existing specialized formulations. Third, we perform a computational study on a large set of randomly generated instances, showing that our model, when solved with a state-of-the-art MIP solver, can obtain optimal solutions in shorter computation times than the specialized formulations.

The paper is organized as follows. In Section 2, we present a survey of the relevant literature. Section 3 introduces a linear MIP formulation for the DFLPG and shows how this model can be used to represent two important special cases. To compare the resulting models with alternative formulations, Section 4 derives specialized formulations for the two special cases, based on existing models from the literature. We identify a weak point in one of the existing formulations and suggest a set of valid inequalities to make it as strong as our model. Dominance relations are proved between all formulations, showing that our model is at least as strong as each of the specialized formulations. The presented models are then compared by means of computational experiments in Section 5. Finally, conclusions follow in Section 6.

# 2 Literature Review

Most dynamic facility location problems can be seen as multi-periodic extensions of classical location problems, such as the Capacitated Facility Location Problem (CFLP). However, dynamic facility location problems commonly involve further extensions. As pointed out by Arabani and Farahani (2011), the notion of what dynamic means may differ when dealing with different areas of facility location. Its definition thus strongly depends on the application context. Owen and Daskin (1998) review works that treat either dynamic or stochastic facility location problems. A chapter in the textbook of Farahani and Hekmatfar (2009) deals with dynamic aspects of facility location problems. Several classification criteria are proposed. A chapter dedicated to multi-period capacitated location models in the textbook of Mirchandani and Francis (1990) thoroughly discusses models that allow capacity expansion. Luss (1982) focuses on capacity expansion and reviews the literature and applications in the context of problems with a single facility, two facilities and multiple facilities. Although not explicitly focusing on dynamic aspects, many other works introduced classifications for location problems which often also apply to features that can be found in dynamic location problems. These include, among many others, the works of Hamacher and Nickel (1998), Owen and Daskin (1998), Klose and Drexl (2005), Daskin (2008) and Melo et al. (2009).

The choice of the facility type or size has also been considered in several works. In particular, Shulman (1991), Correia and Captivo (2003) and Troncoso and Garrido (2005) consider such choice, which implies different capacities and costs for each facility type. The last authors apply the model to the forestry sector, where facilities of different sizes may also be expanded. Dias et al. (2007) focus on modular capacity expansion and reduction. Wu et al. (2006) present a facility location problem, where the facility set-up costs depend on the number of facilities placed at a site. To represent economies of scale, all of the cited works use binary variables to represent the choice of the facility size. Capacity level changes consider only the amount of capacity added or removed. However, the previous capacity level is not taken into consideration. Some authors such as Harkness (2003) also recognize the importance of inverse economies of scale, where the unit price increases as the facility gets larger.

To dynamically adjust capacity to demand changes, the best choice depends on the demand forecast. When demand temporarily decreases, but is likely to return to its previous level afterwards, it may be beneficial to avoid high maintenance costs by temporary closing a facility. The closing and reopening of facilities may be partial or complete. Previous studies focused mostly on temporarily closing entire facilities. Among the suggested models, certain are limited to a single closing and reopening of each facility, whereas others allow repeated closing and reopening throughout the planning horizon. The uncapacitated facility location problem presented by Van Roy and Erlenkotter (1982), as well as the supply chain model of Hinojosa et al. (2008), allow one-time opening or closing of facilities: new facilities can be opened once and existing facilities can be closed once. Chardaire et al. (1996) and Canel et al. (2001) propose formulations for opening and closing facilities more than once. Both works use binary variables to represent the state of the facility. The objective function contains a bilinear term to represent a state change from open to closed or vice-versa. A linear formulation for a simplified version of this problem, treating only a single capacity level, has been proposed by Dias et al. (2006). Binary variables with two time indices indicate the period throughout which a facility is open. The cited works interpret facility closing either as temporary (i.e., the facility still exists, but its capacities are temporarily unavailable) or permanent. In most of the cases, there are no maintenance costs for temporarily closed facilities. Most of the existing formulations therefore do not explicitly distinguish temporary and permanent facility closing.

When the customer demand permanently changes in a certain region and it is not likely to return to its previous level, one may want to expand or reduce the facility capacities to permanently adjust to these new conditions. Luss (1982) observes that models for capacity expansion can be classified into two categories: capacity expansion at a single facility and capacity expansion via a finite set of projects, each holding a certain capacity. The first category includes models that allow one facility at a location and increases or decreases the available capacity along time. The second category consists of models where multiple facilities are allowed in the same location, each specified by a time interval (a capacity block) of production availability. Figure 1 illustrates both classes. The first class is shown in (a), where capacities at the same facility are either increased or decreased. The second class may be illustrated by (b) and (c), representing two extreme configurations of the capacity blocks. Any configuration between these two is also feasible for the second class.

Models in the first category include those of Melo et al. (2005) and Behmardi and Lee (2008). Both works model capacity expansion and reduction by relocating capacity from or to a fictional location. The authors of the former work model capacities as a continuous flow, but demonstrate how to link the flow to binary variables to restrict capacity changes to modular sizes. Models in the second category do not allow the capacity modification of a facility once it is constructed. However, they allow multiple facilities of different sizes (capacity blocks) at the same location, which is equivalent to the adjustment of the total capacity

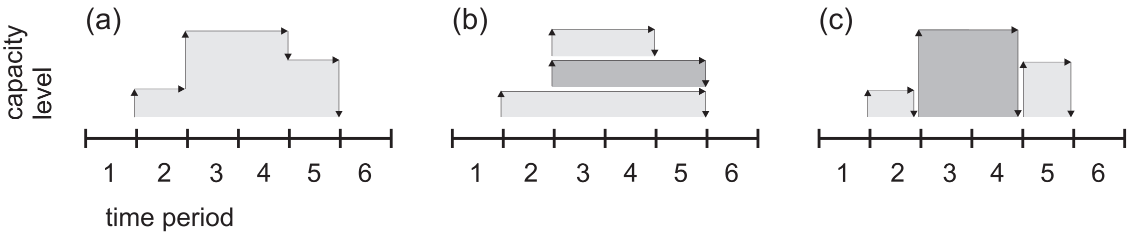


Figure 1: Capacity expansion/reduction by use of a single facility (a), horizontal capacity blocks (b) and vertical capacity blocks (c).

sum along time. Examples for this class include the works of Shulman (1991), Troncoso and Garrido (2005) and Dias et al. (2007). More restricted types of capacity expansion or reduction have also been presented. In the work of Peeters and Antunes (2001), either a facility expands its capacity throughout the entire planning horizon or it decreases its capacity throughout the entire planning horizon. Capacity expansion and reduction at the same location is thus not allowed.

# 3 Mathematical Formulation

In this section, we give a more formal description of the DFLPG and we introduce a MIP model for the problem. We also explain how the different cases described in Section 2 can be modeled and solved as a DFLPG.

## 3.1 DFLPG Formulation

We denote by $J$ the set of potential facility locations and by $L = \{0, 1, 2, \ldots, q\}$ the set of possible capacity levels for each facility. We also denote by $I$ the set of customer demand points and by $T = \{1, 2, \ldots, |T|\}$ the set of time periods in the planning horizon. We assume throughout that the beginning of period $t + 1$ corresponds to the end of period $t$. The demand of customer $i$ in period $t$ is denoted by $d_{it}$, while the cost to serve one unit from facility $j$ operating at capacity level $\ell$ to customer $i$ during period $t$ is denoted by $g_{ij\ell t}$. The capacity of a facility of size $\ell$ at location $j$ is given by $u_{j\ell}$ (with $u_{j0} = 0$). The cost matrix $f_{j\ell_1\ell_2 t}$ describes the combined cost to change the capacity level of a facility at location $j$ from $\ell_1$ to $\ell_2$ at the beginning of period $t$ and to operate the facility at capacity level $\ell_2$ throughout time period $t$. Furthermore, we let $\ell^j$ be the capacity level of an existing facility at location $j$.

To formulate the problem, we use binary variables $y_{j\ell_1\ell_2 t}$ equal to 1 if and only if the facility at location $j$ changes its capacity level from $\ell_1$ to $\ell_2$ at the beginning of period $t$. The allocation variables $x_{ij\ell t}$ denote the fraction of the demand of customer $i$ in period $t$ that is served from a facility of size $\ell$ located at $j$. Based on these definitions, we define the following MIP model, referred to as the *Generalized Modular Capacities*

*(GMC)* formulation:

$$\text{(GMC)} \quad \min \sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L}\sum_{t\in T} f_{j\ell_1\ell_2 t} y_{j\ell_1\ell_2 t} + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} g_{ij\ell t} d_{it} x_{ij\ell t} \tag{1}$$

$$s.t. \quad \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell t} = 1 \quad \forall i \in I, \ \ \forall t \in T \tag{2}$$

$$\sum_{i\in I} d_{it} x_{ij\ell t} \leq \sum_{\ell_1\in L} u_{j\ell} y_{j\ell_1\ell t} \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T \tag{3}$$

$$\sum_{\ell_1\in L} y_{j\ell_1\ell(t-1)} = \sum_{\ell_2\in L} y_{j\ell\ell_2 t} \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T \setminus \{1\} \tag{4}$$

$$\sum_{\ell_2\in L} y_{j\ell^j\ell_2 1} = 1 \quad \forall j \in J \tag{5}$$

$$x_{ij\ell t} \geq 0 \quad \forall i \in I, \ \ \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T \tag{6}$$

$$y_{j\ell_1\ell_2 t} \in \{0,1\} \quad \forall j \in J, \ \ \forall \ell_1 \in L, \ \ \forall \ell_2 \in L, \ \ \forall t \in T. \tag{7}$$

The objective function (1) minimizes the total cost for changing the capacity levels and allocating the demand. Constraints (2) are the demand constraints for the customers. Constraints (3) are the capacity constraints at the facilities. Constraints (4) link the capacity change variables in consecutive time periods. Finally, constraints (5) specify that exactly one capacity level must be chosen at the beginning of the planning horizon. Note that the flow constraints (4) further guarantee that, at each time period, exactly one capacity change variable is selected.

**Valid Inequalities.** To facilitate the solution of the GMC, we may additionally use two types of valid inequalities. The *Strong Inequalities (SI)* used in facility location and network design problems (see, for instance, Gendron and Crainic, 1994) are known to provide a tight upper bound for the demand assignment variables. These inequalities can be adapted to our model as follows:

$$x_{ij\ell t} \leq \sum_{\ell_1\in L} y_{j\ell_1\ell t} \quad \forall i \in I, \ \ \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T. \tag{8}$$

The SIs may be added to the model either *a priori* or in a branch-and-cut manner only when they are violated in the solution of the LP relaxation. The second set of valid inequalities is referred to as the *Aggregated Demand Constraints (ADC)*. Although they are redundant for the LP relaxation, adding them to the model enables MIP solvers to generate cover cuts that further strengthen the formulation:

$$\sum_{j\in J}\sum_{\ell_1\in L}\sum_{\ell_2\in L} u_{j\ell_2} y_{j\ell_1\ell_2 t} \geq \sum_{i\in I} d_{it} \quad \forall t \in T. \tag{9}$$

## 3.2 DFLPG Based Models for the Special Cases

We now explain how two important special cases can be modeled with the GMC formulation: first, *Facility closing and reopening* and, second, *Capacity expansion and reduction.*

The first problem considered here allows the construction of at most one facility per location. The size of the facility is chosen from a discrete set of capacity levels. Existing facilities may be closed and reopened multiple times. Note that, in this problem, facility closing does not refer to permanent closing, but only to the temporary closing of a facility. We therefore distinguish costs for the construction of a facility, for temporarily closing an open facility, for reopening a closed facility and for maintenance of open facilities. As most of the previous literature, we do not consider maintenance costs for temporarily closed facilities. We denote this problem as the *Dynamic Modular Capacitated Facility Location Problem with Closing and Reopening (DMCFLP_CR).*

In the second problem considered, production capacities can be adjusted by the use of a single facility at each location. At each facility, the capacity can be expanded or reduced from one capacity level to another. We assume that an expansion of $\ell$ capacity levels has always the same costs, regardless of the previous capacity level. We assume the same for the reduction of capacities. We denote this problem as the *Dynamic Modular Capacitated Facility Location Problem with Capacity Expansion and Reduction (DMCFLP_ER).*

In addition to the input data already defined for the DFLPG, we define the following parameters to characterize these two special cases:

- $c^c_{j\ell}$ and $c^o_{j\ell}$ are the costs to temporarily close and reopen a facility of size $\ell$, respectively;
- $f^c_{j\ell}$ and $f^o_{j\ell}$ are the costs to reduce and to expand the capacity of a facility at location $j$ by $\ell$ capacity levels, respectively;
- $F^o_{j\ell}$ is the cost to maintain an open facility of size $\ell$ throughout one time period.

For the sake of simplicity and without loss of generality, we assume that all these costs do not change during the planning horizon.

In the GMC, capacity level changes are represented by the $y_{j\ell_1\ell_2t}$ variables. These transitions from one capacity level to another can be represented in a graph, where each node represents a capacity level and each arc a capacity level transition. To model the special cases, we choose a certain subset of arcs, as well as their corresponding objective function coefficients $f_{j\ell_1\ell_2t}$. For the problem variant involving facility closing and reopening, we create an artificial capacity level $\bar{\ell}$ for each capacity level $\ell \in L\backslash\{0\}$. Capacity level $\bar{\ell}$ represents the state in which a facility of size $\ell$ is temporarily closed. At each time period $t \in T$ and location $j \in J$, we may find different arc types $y_{j\ell_1\ell_2t}$ to model capacity level changes (note that the cost for an arc is usually composed by the cost to perform the capacity transition, as well as the maintenance costs for the new capacity level):

1. Facility construction and capacity expansion. The expansion of the capacity is represented by an arc from capacity level $\ell_1$ to any other capacity level $\ell_2 > \ell_1$. If the arc represents a facility construction, then $\ell_1$ is 0. The capacity is thus expanded by $\ell_2 - \ell_1$ capacity levels. The cost for this arc is set to $f_{j\ell_1\ell_2t} = f^o_{j(\ell_2-\ell_1)} + F^o_{j\ell_2}$.
2. Capacity reduction. The reduction of the capacity is represented by an arc from capacity level $\ell_1$ to any other capacity level $\ell_2 < \ell_1$. The capacity is thus reduced by $\ell_1 - \ell_2$ capacity levels. The cost for this arc is set to $f_{j\ell_1\ell_2t} = f^c_{j(\ell_1-\ell_2)} + F^o_{j\ell_2}$.
3. Maintaining the current capacity level. A facility may neither expand nor reduce the current capacity level. The cost of this arc is thus only composed of the maintenance cost, i.e., $f_{j\ell_1\ell_1t} = F^o_{j\ell_1}$ if the capacity level represents an open facility, $f_{j\bar{\ell}_1\bar{\ell}_1t} = 0$ if the capacity level represents a temporarily closed facility and $f_{j00t} = 0$ if no facility exists.
4. Temporary closing. An open facility of size $\ell$ can be temporarily closed, i.e., it changes to capacity level $\bar{\ell}$. The total cost is $f_{j\ell_1\bar{\ell}_1t} = c^c_{j\ell_1}$.
5. Reopening a closed facility. A temporarily closed facility of size $\ell$ can be reopened, i.e., it changes its capacity level from $\bar{\ell}$ to $\ell$. The total cost for this arc is $f_{j\bar{\ell}_1\ell_1t} = c^o_{j\ell_1} + F^o_{j\ell_1}$.

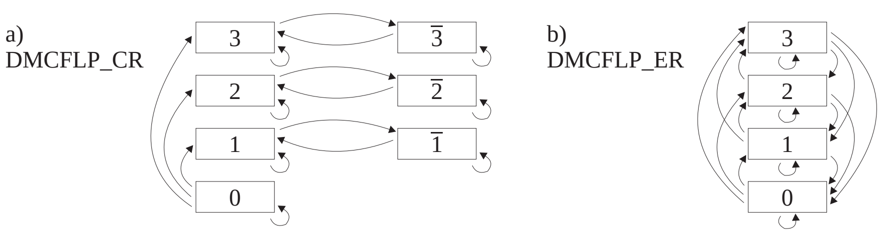


Figure 2: Capacity level changes to model the two special cases.

The DMCFLP_CR is represented by arcs of type 1 (for construction only), 3, 4 and 5. We denote the resulting model as the *CR-GMC* formulation. The resulting graph is shown in Figure 2 (a) for the case of three capacity levels. The DMCFLP_ER is represented by arcs of type 1, 2 and 3. The resulting model is denoted as the *ER-GMC* formulation. Its resulting graph is shown in Figure 2 (b).

# 4 Comparisons with Specialized Formulations

We now present alternative formulations for the two special cases discussed in Section 3.2. These formulations are adaptations of existing models previously proposed in the literature. For each problem, we present formulations based on two different modeling approaches as presented in Section 2: location variables with one time index and location variables with two time indices.

## 4.1 Facility Closing and Reopening

We consider models for the problem with facility closing and reopening, the DMCFLP_CR.

### 4.1.1 Single Time Index Flow Formulation

This model can be seen as an extension of existing dynamic facility location problems (Shulman, 1991). Flow conservation constraints such as those used in the relocation model of Wesolowsky and Truscott (1975) are adapted to model facility closing and reopening. The model is based on the following variables. The demand allocation from facilities to customers is given by $x_{ij\ell t}$. Binary variable $s_{j\ell t}$ is 1 if a facility of size $\ell$ is constructed at the beginning of period $t$, while binary flow variable $y_{j\ell t}$ indicates whether a facility of size $\ell$ is available at location $j$ during time period $t$. Finally, binary variables $v^o_{j\ell t}$ and $v^c_{j\ell t}$ are equal to 1 if an temporarily closed facility at location $j$ of size $\ell$ is reopened at the beginning of period $t$ and if an open facility at location $j$ of size $\ell$ is temporarily closed at the beginning of period $t$, respectively. The input data is as defined in Section 3.2. The *single time index flow formulation (CR-1I)* is given by:

$$\text{(CR-1I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t} + F^o_{j\ell}y_{j\ell t} + c^o_{j\ell}v^o_{j\ell t} + c^c_{j\ell}v^c_{j\ell t}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} d_{it}g_{ij\ell t}x_{ij\ell t} \tag{10}$$

$$s.t. \quad \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell t} = 1 \quad \forall i\in I,\ \ \forall t\in T \tag{11}$$

$$\sum_{i\in I} d_{it}x_{ij\ell t} \le u_{j\ell}y_{j\ell t} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{12}$$

$$y_{j\ell t} = y_{j\ell(t-1)} + s_{j\ell t} + v^o_{j\ell t} - v^c_{j\ell t} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{13}$$

$$\sum_{t'=1}^{t} v^o_{j\ell t'} \le \sum_{t'=1}^{t} v^c_{j\ell t'} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{14}$$

$$\sum_{\ell\in L}\sum_{t\in T} s_{j\ell t} \le 1 \quad \forall j\in J \tag{15}$$

$$x_{ij\ell t} \ge 0 \quad \forall i\in I,\ \ \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{16}$$

$$s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}, y_{j\ell t} \in \{0,1\} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T. \tag{17}$$

Constraints (11) are the demand constraints. Constraints (12) are the capacity constraints. The flow constraints (13) manage the state of a facility of a certain size, either open or closed. Constraints (14) ensure that a facility has to be temporarily closed before it can be reopened. Finally, constraints (15) state that at most one facility can be constructed at each location.

The Strong Inequalities (8) can be adapted by replacing the right hand side by $y_{j\ell t}$, while the Aggregated Demand Constraints (9) can be used by replacing the left hand side by $\sum_{j\in J}\sum_{\ell\in L} u_{j\ell}y_{j\ell t}$.

### 4.1.2 Double Time Index Block Formulations

Dias et al. (2006) presented a linear MIP model that allows the repeated closing and reopening of facilities. The model uses binary decision variables with two time indices, one for the opening and one for the closing of a facility. We extend this model by adding the choice of different facility capacity levels (note that we remove the constraints that require a minimum availability of open facilities). We also use a different notation to be consistent with our previously introduced notations. Binary variable $s_{j\ell t_1 t_2}$ is 1 if a facility of size $\ell$ is constructed at location $j$ at the beginning of time period $t_1$ and stays open until the end of period $t_2$. Binary variable $y_{j\ell t_1 t_2}$ is 1 if an existing facility of size $\ell$, located at $j$, is reopened at the beginning of time period $t_1$ and stays open until the end of period $t_2$. We let $\hat{f}^C_{j\ell t_1 t_2}$ denote the aggregated cost to construct a facility of size $\ell$ at location $j$ at time period $t_1$, its maintenance costs from the beginning of period $t_1$ to the end of period $t_2$, and the costs to temporarily close it at the end of period $t_2$. We also let $\hat{f}^R_{j\ell t_1 t_2}$ denote the same type of cost for reopening an existing facility of size $\ell$ instead of its construction. These constants are computed as follows:

$$\hat{f}^C_{j\ell t_1 t_2} = f^o_{j\ell} + c^c_{j\ell} + (t_2 - t_1 + 1)F^o_{j\ell} \quad \text{and} \quad \hat{f}^R_{j\ell t_1 t_2} = c^o_{j\ell} + c^c_{j\ell} + (t_2 - t_1 + 1)F^o_{j\ell}.$$

Since the binary variables with two time indices describe capacity blocks through time, we refer to this formulation as the *double time index block formulation (CR-2I)*:

$$\text{(CR-2I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t_1\in T}\sum_{t_2=t_1}^{|T|} \left(\hat{f}^C_{j\ell t_1 t_2} s_{j\ell t_1 t_2} + \hat{f}^R_{j\ell t_1 t_2} y_{j\ell t_1 t_2}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} g_{ij\ell t} d_{it} x_{ij\ell t} \tag{18}$$

$$s.t. \quad \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell t} = 1 \quad \forall i \in I, \ \ \forall t \in T \tag{19}$$

$$\sum_{t_2=t}^{|T|} y_{j\ell t t_2} \le \sum_{t_1=1}^{t-1}\sum_{t_2=t_1}^{t-1} s_{j\ell t_1 t_2} \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T \tag{20}$$

$$\sum_{\ell\in L}\sum_{t_1\in T}\sum_{t_2=t_1}^{|T|} s_{j\ell t_1 t_2} \le 1 \quad \forall j \in J \tag{21}$$

$$\sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} (s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}) \le 1 \quad \forall j \in J, \ \ \forall t \in T \tag{22}$$

$$\sum_{i\in I} d_{it} x_{ij\ell t} \le \sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} u_{j\ell}(s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}) \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T \tag{23}$$

$$x_{ij\ell t} \ge 0 \quad \forall i \in I, \ \ \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T \tag{24}$$

$$s_{j\ell t_1 t_2}, y_{j\ell t_1 t_2} \in \{0, 1\} \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t_1 \in T, \ \ \forall t_2 \in T. \tag{25}$$

Constraints (19) are the demand constraints. Constraints (20) guarantee that a facility can only be reopened if it has been constructed and temporarily closed in an earlier period. Inequalities (21) impose that a facility can be constructed only once throughout the entire planning horizon. Constraints (22) guarantee that the intervals of open facilities (i.e., the capacity blocks) at the same location do not intersect. In other words, a facility can only be reopened if it is currently closed. In addition, these constraints also require that only one facility size $\ell$ is selected at each location. Constraints (23) are the facility capacity constraints.

The Strong Inequalities (8) can be adapted by replacing the right hand side by $\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2}$. The Aggregated Demand Constraints (9) can be used by replacing the left hand side by $\sum_{j\in J}\sum_{\ell\in L}\sum_{t_1\in T}\sum_{t_2=t_1}^{|T|} u_{j\ell}(s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2})$.

**Strengthening the CR-2I formulation.** Constraints (20) specify that, at each time period $t$, the capacity that is reopened at this period cannot be greater than the capacity that has been previously constructed. Consider the following LP relaxation solution scenario, where demands exist for three time periods $t_1$, $t_2$ and $t_3$. A facility construction variable is selected with solution value 0.5, opening at the beginning of $t_1$ and closed at the end of $t_1$ (i.e., $y_{j\ell t_1 t_1} = 0.5$). Facility reopening variables are then selected twice, each time with the same solution value 0.5. The first reopening spans the time interval from the beginning of $t_2$ until the end of $t_3$ (i.e., $y_{j\ell t_2 t_3} = 0.5$), whereas the second reopening spans the time interval from the beginning of $t_3$ until the end of $t_3$ (i.e., $y_{j\ell t_3 t_3} = 0.5$). Separately, each of the last two reopenings is feasible in constraints (20). However, in total the solution reopens more capacity than has been made available through construction. To avoid such behaviour in the LP relaxation solution, we may replace constraints (20) with the tighter set of constraints:

$$\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} y_{j\ell t_1 t_2} \leq \sum_{t_1=1}^{t}\sum_{t_2=t_1}^{t} s_{j\ell t_1 t_2} \quad \forall j \in J,\ \ \forall \ell \in L,\ \ \forall t \in T. \tag{26}$$

We denote the formulation composed by (18) - (26) as the CR-2I+ formulation.

#### 4.1.3 Dominance Relationships

For any integer linear programming model $P$, let $\overline{P}$ be the corresponding LP relaxation. For any model $P$, we denote by $v(P)$ its optimal value. For the three models presented for the DMCFLP_CR, the following relationships hold:

**Theorem 4.1** $v(\overline{CR\text{-}GMC}) = v(\overline{CR\text{-}1I}) \geq v(\overline{CR\text{-}2I})$.

Proof. See Appendices A.1.1 and A.1.2. If the strengthening constraints (26) are added to the CR-2I formulation, all formulations are equally strong:

**Theorem 4.2** $v(\overline{CR\text{-}GMC}) = v(\overline{CR\text{-}1I}) = v(\overline{CR\text{-}2I+})$.

Proof. See Appendices A.1.3 and A.1.3.

### 4.2 Capacity Expansion and Reduction

We consider models for the facility location problem with capacity expansion and reduction, the DM-CFLP_ER.

#### 4.2.1 Single Time Index Flow Formulation

We modify the CR-1I as follows. Binary variables $s_{j\ell t}$ now represent the total capacity expansion. A variable $s_{j\ell t}$ is 1 if the capacity of the facility located at $j$ is expanded by $\ell$ capacity levels in the beginning of period $t$. Binary variable $w_{j\ell t}$ is 1 if the capacity of a facility located at $j$ is reduced by $\ell$ capacity levels at the beginning of period $t$. We refer to this formulation as the *single time index flow formulation (ER-1I)*:

$$\text{(ER-1I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t} + f^c_{j\ell}w_{j\ell t} + F^o_{j\ell}y_{j\ell t}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} d_{it}g_{ij\ell t}x_{ij\ell t} \tag{27}$$

$$s.t.\ \ (11), (12)$$

$$\sum_{\ell\in L}\ell y_{j\ell t} = \sum_{\ell\in L}\left(\ell y_{j\ell(t-1)} + \ell s_{j\ell t} - \ell w_{j\ell t}\right) \quad \forall j \in J,\ \ \forall t \in T \tag{28}$$

$$x_{ij\ell t} \geq 0 \quad \forall i \in I,\ \ \forall j \in J,\ \ \forall \ell \in L,\ \ \forall t \in T \tag{29}$$

$$s_{j\ell t}, w_{j\ell t}, y_{j\ell t} \in \{0,1\} \quad \forall j \in J,\ \ \forall \ell \in L,\ \ \forall t \in T. \tag{30}$$

Now, the flow conservation constraints (28) manage the size of the facilities throughout the planning periods.

The model may be seen as an adaptation of the relocation model of Wesolowsky and Truscott (1975), where capacity is expanded or reduced instead of relocated. It is also similar to the model presented by Mirchandani and Francis (1990) and to simplifications of the models presented by Melo et al. (2005) and Behmardi and Lee (2008).

If the costs for facility maintenance, capacity expansion and capacity reduction include economies of scale, the optimal solution contains at most one active $y$, $s$ and $w$ variable for each location and time period. However, if the cost structure does not necessarily consider economies of scale, we need to add constraints to ensure that at most one variable of each type is active at each location $j \in J$ and time period $t \in T$. We refer to these constraints as the *limiting constraints*: $\sum_{\ell\in L} y_{j\ell t} \leq 1$ for the $y$ variables, $\sum_{\ell\in L} s_{j\ell t} \leq 1$ for the $s$ variables and $\sum_{\ell\in L} w_{j\ell t} \leq 1$ for the $w$ variables.

### 4.2.2 Double Time Index Block Formulations

Dias et al. (2007) allow multiple capacity blocks of different sizes at the same location. For each block, binary variables define the exact time interval in that the block is active. This accumulation of capacity blocks allows flexible capacity expansion and reduction as previously discussed and exemplified in Figure 1 (b) and (c). We extend this formulation to model the DMCFLP_ER.

Binary variables $y'_{j\ell t_1 t_2}$ indicate whether a capacity block of size $\ell$ is available at location $j$ from the beginning of time period $t_1$ until the end of time period $t_2$. Each capacity block may thus represent economies of scale in function of its own size. However, in contrast to the ER-1I, the total capacity available at a location can now be composed by several capacity blocks. To consider economies of scale on the entire capacity involved at each location, we introduce additional binary variables $y_{j\ell t}$ to represent the total capacity summed over all capacity blocks at location $\ell$ available at time period $t$. In the same manner, we introduce variables $s_{j\ell t}$ and $w_{j\ell t}$ to represent the total capacity that is added at a location (i.e., the construction of capacity blocks) or removed at a location (i.e., the closing of capacity blocks), respectively. Finally, as in the previous models, $x_{ij\ell t}$ is the fraction of customer $i$'s demand that is served by a facility of size $\ell$ at location $j$. The *double time index block formulation (ER-2I)* is given by:

$$\text{(ER-2I)} \quad \min \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T}\left(f^o_{j\ell}s_{j\ell t} + f^c_{j\ell}w_{j\ell t} + F^o_{j\ell}y_{j\ell t}\right) + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} d_{it}g_{ij\ell t}x_{ij\ell t} \tag{31}$$

$$s.t. \quad (11), (12)$$

$$\sum_{\ell\in L}\ell s_{j\ell t} = \sum_{\ell\in L}\sum_{t_2=t}^{|T|}\ell y'_{j\ell t t_2} \quad \forall j\in J,\ \ \forall t\in T \tag{32}$$

$$\sum_{\ell\in L}\ell w_{j\ell t} = \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\ell y'_{j\ell t_1 (t-1)} \quad \forall j\in J,\ \ \forall t\in T \tag{33}$$

$$\sum_{\ell\in L}\ell y_{j\ell t} = \sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|}\ell y'_{j\ell t_1 t_2} \quad \forall j\in J,\ \ \forall t\in T \tag{34}$$

$$x_{ij\ell t} \geq 0 \quad \forall i\in I,\ \ \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{35}$$

$$y'_{j\ell t_1 t_2}, s_{j\ell t}, w_{j\ell t}, y_{j\ell t} \in \{0,1\} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t_1\in T,\ \ \forall t_2\in T. \tag{36}$$

We adapt the demand and capacity constraints (11) and (12), respectively, from the previous models. Constraints (32), (33) and (34) are the linking constraints that set the binary variables to benefit from economies of scale in function of the total capacity involved in each operation and location. As for the ER-1I formulation, we also add the limiting constraints as introduced in Section 4.2. The limiting constraints are necessary to ensure that feasible solutions use only one active variable of each type $y$, $s$ and $w$ for each location and time period. These constraints have also proved to facilitate the solution process. We may also add the Strong Inequalities and the Aggregated Demand Constraints.

### 4.2.3 Dominance Relationships

For the DMCFLP_ER, the ER-GMC formulation is stronger (strictly stronger for some instances) than the other two formulations:

**Theorem 4.3** $v(\overline{ER\text{-}GMC}) \geq v(\overline{ER\text{-}1I}) = v(\overline{ER\text{-}2I})$.

Proof. See Appendices A.2.2 and A.2.1.

## 5 Computational Experiments

In this section, we explain how the problem instances used in the computational experiments were generated. Then, computational results are given to illustrate the strength of the different formulations and their performance when using a state-of-the-art MIP solver to find optimal integer solutions. All computational experiments were performed for the two problem variants, DMCFLP_CR and DMCFLP_ER.

## 5.1 Problem Instances

Instances have been generated to evaluate the performance of the proposed formulations. The key parameters that were found to affect the difficulty of the problem are:

- **Problem dimension**. Instances have been generated with the following dimensions *(#facility locations/#customers)*: *(10/20)*, *(10/50)*, *(50/50)*, *(50/100)*, *(50/250)*, *(100/250)*, *(100/500)* and *(100/1000)*. The problem dimension clearly impacts the size of the models.

- **Number of capacity levels**. The number of capacity levels $q$ also impacts the size of the models. Instances are generated with a maximum of 3, 5 and 10 capacity levels. Within each of these three cases, the difference of the capacity between two capacity levels $\ell$ and $\ell+1$ is constant, i.e. $u_{j\ell} - u_{j(\ell+1)} = u_{j1}$ for all $j \in J$ and $\ell = 0, \ldots, (q-1)$.

- **Distances and transportation costs**. For each of the problem sizes, three different networks have been randomly generated on squares of the following sizes: $300 \times 300$, $380 \times 380$ and $450 \times 450$. Coordinates for customers and facilities have been generated randomly following a uniform distribution. Transportation costs have been computed based on the Euclidean distance between the points.

- **Demand distribution**. Demands are randomly generated. We consider two different demand scenarios. In the first scenario, the total demand summed over all customers is the same in each time period. However, the demand for each customer is randomly distributed over time. The second scenario assumes that the total demand in each period is not stable, but instead follows strong variations. In this distribution, the total demand in each period for all customers is randomly generated. Then, the demand for each customer is randomly generated and distributed over time.

- **Cost distribution**. The proportion between facility construction and transportation costs is generated based on different ratios. The transportation costs were set to 100% and 500% of the original transportation costs.

All generated instances contain ten time periods. Construction and operational costs follow concave cost functions, i.e., they involve economies of scale. Note that we assume that the problem instances do not contain initially existing facilities. The final test set contains a total of 288 instances, 96 for each capacity level.

All mathematical models have been implemented in C/C++ using the IBM CPLEX 12.4.0 Callable Library. The code has been compiled and executed on openSUSE 11.3. Each problem instance has been run on a single Intel Xeon X5650 processor (2.67GHz), limited to 24GB of RAM.

## 5.2 Linear Relaxation Solution and Integrality Gaps

We now compare the different formulations for the two problem variants by means of their LP relaxation bounds as well as the time necessary to solve the LP relaxations. All SIs have been added *a priori*. The Aggregated Demand Constraints have not been added to these models, since they do not have any impact on the strength of the LP relaxation. For all instances, the LP relaxation has been solved to optimality. Table 1 shows the average times to solve the LP relaxation as well as the average integrality gaps, for each problem dimension and each number of maximum capacity levels $q$.

As previously shown, the CR-1I, the CR-2I+ and the CR-GMC formulations provide the same LP relaxation bound and thus the same integrality gap. However, the CR-1I and the CR-GMC formulations solve the relaxation in much shorter computation times than the CR-2I+ formulation.

For the DMCFLP_ER, the ER-1I and ER-2I formulations provide the same integrality gaps. The ER-GMC formulation provides a significantly smaller integrality gap than the previous two formulations, while having similar computation times.

## 5.3 CPLEX Optimization

We now compare the performance of the different formulations to find optimal integer solutions. We used the MIP branch-and-cut algorithm of CPLEX 12.4.0 with standard parameters. Computation times have

| | | DMCFLP_CR | | | | DMCFLP_ER | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | ER-1I | | ER-2I | | ER-GMC | |
| q | Instance size | Time (sec) 1I | 2I+ | GMC | Integr. Gap % | Time (sec) | Integr. Gap % | Time (sec) | Integr. Gap % | Time (sec) | Integr. Gap % |
| 3 | 10/20 | 0.2 | 0.3 | 0.0 | 0.90 | 0.0 | 2.29 | 0.0 | 2.29 | 0.1 | 0.83 |
| | 10/50 | 0.0 | 0.2 | 0.1 | 0.18 | 0.0 | 0.79 | 0.0 | 0.79 | 0.0 | 0.21 |
| | 50/50 | 0.3 | 0.8 | 0.2 | 0.09 | 0.3 | 2.32 | 0.1 | 2.32 | 0.5 | 0.14 |
| | 50/100 | 0.5 | 1.5 | 0.6 | 0.02 | 0.4 | 1.13 | 0.8 | 1.13 | 0.6 | 0.00 |
| | 50/250 | 1.2 | 2.8 | 1.0 | 0.00 | 0.7 | 0.62 | 1.1 | 0.62 | 1.3 | 0.00 |
| | 100/250 | 4.3 | 8.3 | 4.5 | 0.01 | 2.8 | 0.93 | 3.6 | 0.93 | 3.9 | 0.01 |
| | 100/500 | 11.2 | 20.0 | 11.3 | 0.03 | 7.8 | 0.60 | 9.3 | 0.60 | 8.8 | 0.02 |
| | 100/1000 | 16.0 | 29.8 | 17.4 | 0.00 | 13.6 | 0.42 | 12.6 | 0.42 | 14.3 | 0.00 |
| | **Avg All** | **4.2** | **8.0** | **4.4** | **0.15** | **3.2** | **1.14** | **3.4** | **1.14** | **3.7** | **0.15** |
| 5 | 10/20 | 1.6 | 0.2 | 0.3 | 1.85 | 0.9 | 4.67 | 0.2 | 4.67 | 0.2 | 1.45 |
| | 10/50 | 0.2 | 0.4 | 0.3 | 0.70 | 0.3 | 2.05 | 0.1 | 2.05 | 0.2 | 0.64 |
| | 50/50 | 1.8 | 3.1 | 1.9 | 0.67 | 2.0 | 5.09 | 1.7 | 5.09 | 1.8 | 0.56 |
| | 50/100 | 1.8 | 4.1 | 2.0 | 0.08 | 1.8 | 2.30 | 1.7 | 2.30 | 2.1 | 0.06 |
| | 50/250 | 2.3 | 5.1 | 2.7 | 0.01 | 4.0 | 1.20 | 2.5 | 1.20 | 2.7 | 0.00 |
| | 100/250 | 8.4 | 18.2 | 9.2 | 0.02 | 8.4 | 1.84 | 8.2 | 1.84 | 8.5 | 0.01 |
| | 100/500 | 16.5 | 35.2 | 17.7 | 0.03 | 16.6 | 1.15 | 18.0 | 1.15 | 16.1 | 0.02 |
| | 100/1000 | 21.8 | 51.3 | 23.6 | 0.00 | 31.3 | 0.80 | 30.5 | 0.80 | 27.5 | 0.00 |
| | **Avg All** | **6.8** | **14.7** | **7.2** | **0.42** | **8.2** | **2.39** | **7.8** | **2.39** | **7.4** | **0.34** |
| 10 | 10/20 | 2.3 | 1.4 | 1.0 | 2.89 | 0.3 | 9.63 | 0.5 | 9.63 | 1.2 | 1.89 |
| | 10/50 | 2.7 | 3.5 | 2.1 | 1.46 | 0.8 | 4.92 | 0.7 | 4.92 | 1.6 | 0.88 |
| | 50/50 | 19.0 | 27.8 | 15.9 | 2.47 | 9.8 | 14.11 | 10.3 | 14.11 | 30.9 | 2.11 |
| | 50/100 | 29.1 | 52.1 | 25.1 | 0.72 | 16.7 | 6.19 | 17.4 | 6.19 | 26.7 | 0.66 |
| | 50/250 | 22.8 | 63.3 | 24.8 | 0.20 | 22.1 | 2.91 | 16.8 | 2.91 | 24.8 | 0.19 |
| | 100/250 | 185.2 | 382.9 | 169.8 | 0.21 | 115.6 | 4.41 | 122.1 | 4.41 | 196.7 | 0.26 |
| | 100/500 | 91.8 | 182.6 | 94.8 | 0.03 | 88.8 | 2.40 | 83.7 | 2.40 | 80.7 | 0.04 |
| | 100/1000 | 51.8 | 123.8 | 56.2 | 0.00 | 55.8 | 1.61 | 62.9 | 1.61 | 64.2 | 0.00 |
| | **Avg All** | **50.6** | **104.7** | **48.7** | **1.00** | **38.7** | **5.77** | **39.3** | **5.77** | **53.3** | **0.75** |

Table 1: LP relaxation and integrality gaps for all formulations.

been limited to six hours. In all CPLEX experiments, all Strong Inequalities have been added *a priori* to the models. Even though the number of SIs may increase significantly, adding them *a priori* (instead of as *CPLEX user cuts* or even not at all) significantly facilitates the solution of the problems. Experiments showed that, for most of the problem instances, a large number of SIs are violated. CPLEX thus spends much time identifying and adding violated SIs when treated as *CPLEX user cuts*. Although redundant to the LP relaxation of the presented formulations, the Aggregated Demand Constraints tend to slightly facilitate the solution of the problems. Therefore, they also have been added to the formulations. For some models, the limiting constraints as shown in Section 4.2 may not change the set of feasible integer solutions, but still facilitate the solution of the problem. For example, for the ER-1I formulation, the average solution time for our test instances decreased by around 35%. The constraints are thus added to the models even if they are redundant.

For each problem, the results have been separated into two groups: instances that have been solved to optimality by all formulations and instances where at least one formulation could not prove optimality within the given time limit. Table 2 summarizes the results for the instances that have been solved by all formulations for each problem. The table reports the number of instances that have been solved to optimality, as well as the average computation times to solve the instances for each of the formulations. For both problem variants, we observe that the 2I formulation performs worst. Among the 1I and the GMC based formulations, the GMC based models provide substantially better results.

Tables 3 and 4 summarize the results for instances where at least one of the formulations did not solve the instance in the given time limit. The table reports average and maximum optimality gaps as well as the number of instances where no feasible solution has been found ($\#ns$). For $q = 5$, one particular instance has been found to be difficult to solve. All other instances are for $q = 10$. The average optimality gaps provided by the GMC based models are always better than those provided by the other formulations. The maximum optimality gap is similar for the 1I and GMC based formulations and worse for the 2I formulations.

**CPLEX root node solution.** For the problem variant involving capacity expansion and reduction, the DMCFLP_ER, the GMC formulation provides much stronger LP relaxation bounds than the 1I and 2I formulations. While the integrality gap of the GMC solution is only a fraction of those provided by the 1I and 2I formulations, this advantage is not translated into the optimization results. Even though the GMC formulation is still faster than the other two formulations, the difference is much smaller. The main reason for that is CPLEX strong cut generation at the root node. Table 5 summarizes the average values (for $q = 10$) regarding the integrality gaps after the LP relaxation and after the root node solution, as well as the standard CPLEX optimization (with cut generation at the root node). The average values are given for all instances that have been solved to optimality. In addition, the last three columns of the table indicate the results obtained by CPLEX when no cuts are generated at the root node. As the 1I and 2I formulations had difficulties to find optimal solutions within the given time limit, these average values include the results for all instances, solved and unsolved. After the LP relaxation, the advantage of a 0.75% integrality gap for the ER-GMC formulation is obvious when compared to the 6.09% integrality gap for the ER-1I formulation. When turning off CPLEX cut generation at the root node, the optimization results directly translate this advantage into faster computation times to solve the problems. However, after solving the root node (with cut generation), the improved 0.52% average integrality gap competes with a 0.79% average integrality gap. This small difference is then directly translated into the optimization results, where the ER-GMC is only slightly faster than the ER-1I. These results demonstrate the efficiency of the GMC formulation for standard branch-and-bound methods, in particular when the MIP solver cut generation is less elaborated than that of CPLEX.

**Facility Closing and Reopening with Capacity Expansion and Reduction.** The two problem variants treated above consider either facility closing/reopening or capacity expansion/reduction. Experiments have also been performed for a third problem variant combining both features. The problem is modeled by the use of the DFLPG by using the transition arcs for both problems as shown in Section 3.2. Additionally, arcs are added representing combined decisions such as facility reopening with subsequent capacity expansion (in the same time period), as well as capacity reduction with subsequent facility closing. Alternatively, a specialized flow formulation can be used with two types of flow constraints: one to manage the capacity of open facilities and one to manage the capacity of closed facilities. The observations made above regarding

| q | Instance size | # Inst | DMCFLP_CR CR-1I | CR-2I+ | CR-GMC | # Inst | DMCFLP_ER ER-1I | ER-2I+ | ER-GMC |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 10/20 | 12 | 0.5 | 1.3 | 0.3 | 12 | 0.1 | 0.5 | 0.3 |
| | 10/50 | 12 | 0.6 | 1.6 | 0.3 | 12 | 0.3 | 0.7 | 0.6 |
| | 50/50 | 12 | 3.0 | 6.9 | 2.4 | 12 | 4.3 | 9.0 | 4.3 |
| | 50/100 | 12 | 3.3 | 10.5 | 3.6 | 12 | 3.5 | 5.3 | 2.6 |
| | 50/250 | 12 | 5.8 | 23.9 | 6.6 | 12 | 9.7 | 9.3 | 6.4 |
| | 100/250 | 12 | 17.9 | 69.3 | 25.0 | 12 | 24.3 | 32.3 | 23.1 |
| | 100/500 | 12 | 67.4 | 307.4 | 100.8 | 11 | 68.0 | 100.8 | 73.4 |
| | 100/1000 | 12 | 64.4 | 310.3 | 66.8 | 12 | 101.7 | 125.7 | 82.0 |
| | **Avg All** | **96** | **20.4** | **91.4** | **25.7** | **95** | **26.0** | **35.4** | **23.5** |
| 5 | 10/20 | 12 | 7.9 | 62.3 | 4.6 | 12 | 2.3 | 17.1 | 1.4 |
| | 10/50 | 12 | 6.4 | 66.2 | 5.7 | 12 | 3.2 | 17.3 | 2.5 |
| | 50/50 | 11 | 184.2 | 130.2 | 37.4 | 11 | 57.0 | 99.3 | 33.0 |
| | 50/100 | 12 | 25.8 | 76.3 | 25.7 | 12 | 18.3 | 33.4 | 13.5 |
| | 50/250 | 12 | 14.2 | 51.9 | 16.3 | 12 | 16.8 | 20.3 | 19.4 |
| | 100/250 | 12 | 43.6 | 189.0 | 87.5 | 12 | 71.6 | 90.1 | 58.9 |
| | 100/500 | 12 | 145.8 | 732.7 | 233.3 | 12 | 188.3 | 261.6 | 158.4 |
| | 100/1000 | 12 | 149.1 | 889.8 | 305.0 | 12 | 283.2 | 347.8 | 197.8 |
| | **Avg All** | **95** | **70.9** | **276.3** | **90.0** | **95** | **80.3** | **111.0** | **60.9** |
| 10 | 10/20 | 9 | 982.3 | 3,322.6 | 360.1 | 11 | 1,447.5 | 4,136.1 | 1,019.5 |
| | 10/50 | 4 | 15.5 | 694.5 | 11.0 | 6 | 677.8 | 2,452.3 | 466.8 |
| | 50/50 | 5 | 4,280.0 | 1,599.8 | 71.8 | 10 | 1,945.3 | 3,495.1 | 1,251.8 |
| | 50/100 | 8 | 2,002.8 | 1,111.1 | 1,837.1 | 10 | 520.3 | 1,037.4 | 260.2 |
| | 50/250 | 10 | 897.1 | 3,429.7 | 1,066.2 | 9 | 2,467.6 | 6,708.0 | 2,479.2 |
| | 100/250 | 8 | 1,146.1 | 864.9 | 2,383.4 | 10 | 1,325.8 | 4,903.0 | 528.1 |
| | 100/500 | 7 | 189.3 | 4,035.3 | 866.4 | 7 | 539.6 | 842.3 | 501.3 |
| | 100/1000 | 11 | 406.0 | 2,037.0 | 438.2 | 11 | 1,097.4 | 3,392.4 | 605.5 |
| | **Avg All** | **62** | **1,133.2** | **2,281.3** | **950.9** | **74** | **1,296.7** | **3,488.5** | **904.0** |

Table 2: CPLEX branch-and-bound results for instances solved to optimality by all formulations for each problem.

| $q$ | Instance size | # Inst | CR-1I Gap % Avg | Max | # ns | CR-2I Gap % Avg | Max | # ns | CR-GMC Gap % Avg | Max | # ns |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 50/50 | 1 | 1.73 | 1.73 | 0 | 2.01 | 2.01 | 0 | 1.24 | 1.24 | 0 |
| 10 | 10/20 | 3 | 0.01 | 0.01 | 0 | 0.50 | 0.71 | 0 | 0.01 | 0.01 | 0 |
| | 10/50 | 8 | 0.01 | 0.01 | 0 | 1.21 | 3.33 | 0 | 0.01 | 0.01 | 0 |
| | 50/50 | 7 | 1.63 | 2.74 | 0 | 0.43 | 1.57 | 0 | 1.25 | 2.58 | 0 |
| | 50/100 | 4 | 0.51 | 1.18 | 0 | 0.90 | 1.50 | 0 | 0.42 | 1.25 | 0 |
| | 50/250 | 2 | 0.13 | 0.26 | 0 | 0.04 | 0.08 | 0 | 0.09 | 0.18 | 0 |
| | 100/250 | 4 | 0.27 | 0.83 | 0 | 0.27 | 1.07 | 0 | 0.26 | 0.74 | 0 |
| | 100/500 | 5 | 0.01 | 0.01 | 0 | 1.19 | 1.42 | 2 | 0.01 | 0.01 | 0 |
| | 100/1000 | 1 | 0.00 | 0.00 | 0 | 1.38 | 1.38 | 0 | 0.01 | 0.01 | 0 |
| | **Avg All** | **34** | **0.44** | **2.74** | **0** | **0.75** | **3.33** | **2** | **0.35** | **2.58** | **0** |

Table 3: CPLEX branch-and-bound results for unsolved instances of the DMCFLP_CR.

| q | Instance size | # Inst | ER-1I Gap % Avg | ER-1I Gap % Max | ER-1I # ns | ER-2I Gap % Avg | ER-2I Gap % Max | ER-2I # ns | ER-GMC Gap % Avg | ER-GMC Gap % Max | ER-GMC # ns |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 50/50 | 1 | 0.61 | 0.61 | 0 | 0.77 | 0.77 | 0 | 0.01 | 0.01 | 0 |
| 10 | 10/20 | 1 | 0.01 | 0.01 | 0 | 0.45 | 0.45 | 0 | 0.01 | 0.01 | 0 |
| | 10/50 | 6 | 1.18 | 3.33 | 0 | 1.85 | 4.83 | 0 | 0.80 | 3.55 | 0 |
| | 50/50 | 2 | 1.21 | 1.47 | 0 | 1.59 | 1.76 | 0 | 0.90 | 1.36 | 0 |
| | 50/100 | 2 | 0.48 | 0.56 | 0 | 6.95 | 13.04 | 0 | 0.42 | 0.43 | 0 |
| | 50/250 | 3 | 0.00 | 0.01 | 0 | 3.33 | 6.36 | 1 | 0.00 | 0.01 | 0 |
| | 100/250 | 2 | 0.42 | 0.83 | 0 | 1.40 | 1.53 | 0 | 0.13 | 0.24 | 0 |
| | 100/500 | 5 | 0.63 | 1.88 | 2 | 1.50 | 3.75 | 2 | 0.09 | 0.41 | 0 |
| | 100/1000 | 1 | 0.01 | 0.01 | 0 | 0.56 | 0.56 | 0 | 0.01 | 0.01 | 0 |
| | **Avg All** | **22** | **0.66** | **3.33** | **2** | **2.27** | **13.04** | **3** | **0.37** | **3.55** | **0** |

Table 4: CPLEX branch-and-bound results for unsolved instances of the DMCFLP_ER.

| Formulation | LP Relaxation Integr. Gap % | LP Relaxation Time (sec) | Root Node Integr. Gap % | Root Node Time (sec) | CPLEX w/ root cuts Time (sec) | CPLEX w/o root cuts Gap % | CPLEX w/o root cuts Time (sec) | CPLEX w/o root cuts # ns |
|---|---|---|---|---|---|---|---|---|
| ER-1I | 6.09 | 30.7 | 0.79 | 353.9 | 1,296.7 | 3.07 | 20,219.4 | 3 |
| ER-2I | 6.09 | 32.3 | 1.03 | 695.6 | 3,488.5 | 3.36 | 20,047.6 | 4 |
| ER-GMC | 0.75 | 46.9 | 0.52 | 379.3 | 904.0 | 0.08 | 3,215.8 | 0 |

Table 5: Results for LP relaxation, root node and integer problem ($q = 10$).

the CPLEX root node solution were also confirmed for this more complex problem variant. In addition, the advantage of the GMC model for this variant is even more obvious than what was observed for the DMCFLP_ER. We proved that the GMC based model provides a stronger LP relaxation than the specialized flow formulation. Computationally, the average integrality gap (for $q = 10$) improved from 6.10% to 0.92% when using the GMC based model instead of the specialized formulation. In the CPLEX optimization, the GMC based formulation is, on average, more than twice as fast as the specialized formulation, improving the average computation time from $2,709$ to $1,273$ seconds.

# 6 Conclusions and Future Research

We have introduced a new general facility location problem that unifies several existing multi-period facility location problems. We showed the flexibility of this generalization by focusing on two problem variants: facility closing and reopening and capacity expansion and reduction. In addition, we also reported results on a variant that combines both of these features. For the two first cases, we derived specialized models based on two well-known formulation approaches. We formally proved that, even though our model is more general, it provides LP relaxation bounds as strong as the other formulations for the case of facility closing/reopening and stronger LP relaxation bounds than the formulations for the other two cases. Computational experiments showed that, for the two variants involving capacity expansion and reduction, the integrality gap of our model is up to 6-7 times smaller than the integrality gaps of the specialized formulations. While solving the root node, CPLEX effectively strengthens the specialized, but weaker formulations by adding cuts. However, as the cut generation by CPLEX is not easily reproducible, our model holds many advantages when compared to the alternative formulations: it is more general, stronger and, on average up to twice as fast to find optimal solutions. We also emphasize that CPLEX possesses one of the strongest pre-processing and cut generation methods among all general-purpose MIP solvers. It is very likely that the dominance of our general model over the specialized formulations would be more obvious when using another MIP solver.

The general model may also be used to model other problem variants not addressed in this work, e.g., the uncapacitated location problem with facility opening and closing of Van Roy and Erlenkotter (1982), the closing and reopening model of Chardaire et al. (1996) or the dynamic location problem of Sridharan

(1995). In addition, problem variants that involve capacity changes may benefit from the proposed modeling technique to strengthen the existing models. Problems such as those presented by Shulman (1991) and Correia and Captivo (2003) can be modeled by the DFLPG when adding individual constraints such as minimum production bounds for the facilities. Finally, as the general model is already very strong, it may also be an ideal candidate for decomposition techniques such as Lagrangean relaxation to find good quality solutions in short computation times.

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# A Theoretical Results

## A.1 Theoretical Results for the DMCFLP_CR formulations

We now prove the dominance relationships between the three formulations presented for the DMCFLP_CR. For any integer linear programming model $P$, let $\overline{P}$ be the corresponding LP relaxation. For any model $P$, we denote by $v(P)$ its optimal value.

### A.1.1 CR-GMC and CR-1I are equally strong

We prove that the LP relaxations of the formulations CR-GMC and CR-1I provide the same lower bound.

**Theorem A.1** $v(\overline{\textit{CR-GMC}}) = v(\overline{\textit{CR-1I}})$.

**Proof.** The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{CR-1I}}$ from any feasible solution for $\overline{\text{CR-GMC}}$ and that both solutions have the same objective function value. Then, we show the same, constructing an equivalent and feasible $\overline{\text{CR-GMC}}$ solution from any feasible $\overline{\text{CR-1I}}$ solution.

To facilitate the proof, we first write the CR-GMC in its explicit form as it is defined in Section 3.2. As previously defined, let $L = \{0, 1, 2, .., q\}$ be the set of available capacity levels to define the facility size. In the same manner, let $L' = \left\{\overline{1}, \overline{2}, .., \overline{q}\right\}$ be the set of closed capacity levels:

$$
\begin{aligned}
\text{(CR-GMC)} \quad \min & \sum_{j\in J}\sum_{\ell_2\in L}\sum_{t\in T}(f^o_{j\ell_2} + F^o_{j\ell_2})y_{j(\ell_1=0)\ell_2 t} + \sum_{i\in I}\sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} g_{ij\ell t}d_{it}x_{ij\ell t} \\
& + \sum_{j\in J}\sum_{\ell_1\in L}\sum_{t\in T}(c^c_{j\ell_1 t} + F^c_{j\ell_1 t})y_{j\ell_1\bar{\ell}_1 t} + \sum_{j\in J}\sum_{\ell_1\in L}\sum_{t\in T}(c^o_{j\ell_1 t} + F^o_{j\ell_1})y_{j\bar{\ell}_1\ell_1 t} \\
& + \sum_{j\in J}\sum_{\ell\in L}\sum_{t\in T} F^o_{j\ell}y_{j\ell t}
\end{aligned}
$$

$$
s.t. \quad \sum_{j\in J}\sum_{\ell\in L} x_{ij\ell t} = 1 \quad \forall i\in I,\ \ \forall p\in P,\ \ \forall t\in T \tag{37}
$$

$$
\sum_{i\in I} d_{it}x_{ij\ell t} \le \sum_{\ell_1\in L} u_{j\ell}y_{j\ell_1\ell t} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{38}
$$

$$
y_{j0\ell(t-1)} + y_{j\ell\ell(t-1)} + y_{j\bar{\ell}\ell(t-1)} = y_{j\ell\ell t} + y_{j\ell\bar{\ell}t} \quad \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{39}
$$

$$
y_{j\ell\bar{\ell}(t-1)} + y_{j\bar{\ell}\bar{\ell}(t-1)} = y_{j\bar{\ell}\bar{\ell}t} + y_{j\bar{\ell}\ell t} \quad \forall j\in J,\ \ \forall \bar{\ell}\in L',\ \ \forall t\in T \tag{40}
$$

$$
\sum_{\ell_1\in L\cup L'}\sum_{\ell_2\in L\cup L'} y_{j\ell_1\ell_2 t} \le 1 \quad \forall j\in J,\ \ \forall t\in T \tag{41}
$$

$$
x_{ij\ell t} \ge 0 \quad \forall i\in I,\ \ \forall j\in J,\ \ \forall \ell\in L,\ \ \forall t\in T \tag{42}
$$

$$
y_{j\ell_1\ell_2 t} \in \{0,1\} \quad \forall j\in J,\ \ \forall \ell_1\in L,\ \ \forall \ell_2\in L,\ \ \forall t\in T. \tag{43}
$$

**(A) Construction of a feasible $\overline{\textbf{CR-1I}}$ solution from any $\overline{\textbf{CR-GMC}}$ solution**

Consider any solution $\{x_{ij\ell t}, y_{j\ell_1\ell_2 t}\}$ that is feasible in $\overline{\text{CR-GMC}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CR-1I}}$ and has the same objective function value.

We set the values for the $x_{ij\ell t}$ variables identical to those in the CR-GMC solution. The values for the variables $y_{j\ell t}$, $s_{j\ell t}$, $v^o_{j\ell t}$ and $v^c_{j\ell t}$ are set by establishing the following relations $\forall j\in J, \forall \ell\in L, \forall t\in T$:

$$
s_{j\ell t} = y_{j0kt} \tag{CR.R1}
$$

$$
v^o_{j\ell t} = y_{j\bar{\ell}\ell t} \tag{CR.R2}
$$

$$
v^c_{j\ell t} = y_{j\ell\bar{\ell}t} \tag{CR.R3}
$$

$$
y_{j\ell t} = y_{j0\ell t} + y_{j\ell\ell t} + y_{j\bar{\ell}\ell t}. \tag{CR.R4}
$$

According to these relations and the way the objective function coefficients are composed (see Section 3.2), it can be easily verified that both solutions have the same objective function value. Constraints (11) and (12) are satisfied, as they contain the same variables with the same values in both models. We show that constraints (13) hold by using the relationships defined above:

$$
\begin{aligned}
& & y_{j\ell t} &= y_{j\ell(t-1)} + s_{j\ell t} + v^o_{j\ell t} - v^c_{j\ell t} && (13)\\
&\overset{\text{(CR.R1) - (CR.R4)}}{\Leftrightarrow} & y_{j0\ell t} + y_{j\ell\ell t} + y_{j\bar{\ell}\ell t} &= y_{j0\ell(t-1)} + y_{j\ell\ell(t-1)} + y_{j\bar{\ell}\ell(t-1)} + y_{j0\ell t} + y_{j\bar{\ell}\ell t} - y_{j\ell\bar{\ell}t} && \\
&\overset{\text{cancel } y_{j0\ell t} \text{ \& } y_{j\bar{\ell}\ell t}}{\Leftrightarrow} & y_{j\ell\ell t} &= y_{j0\ell(t-1)} + y_{j\ell\ell(t-1)} + y_{j\bar{\ell}\ell(t-1)} - y_{j\ell\bar{\ell}t}. && (39)
\end{aligned}
$$

As equalities (39) necessarily hold, constraints (13) are also satisfied. In a similar way, we show that constraints (14) hold:

$$
\begin{aligned}
& & \sum_{t'=1}^{t} v^o_{j\ell t'} &\leq \sum_{t'=1}^{t} v^c_{j\ell t'} && (14)\\
&\overset{\text{(CR.R2) \& (CR.R3)}}{\Leftrightarrow} & \sum_{t'=1}^{t} y_{j\bar{\ell}\ell t'} &\leq \sum_{t'=1}^{t} y_{j\ell\bar{\ell}t'} && \\
&\overset{\text{replace LHS by (40)}}{\Leftrightarrow} & \sum_{t'=1}^{t} y_{j\ell\bar{\ell}(t'-1)} + \sum_{t'=1}^{t} y_{j\bar{\ell}\bar{\ell}(t'-1)} + \sum_{t'=1}^{t} y_{j\bar{\ell}\bar{\ell}t'} &\leq \sum_{t'=1}^{t} y_{j\ell\bar{\ell}t'} && \\
&\overset{\text{cancel } y_{j\bar{\ell}\bar{\ell}t} \text{ \& } y_{j\ell\bar{\ell}t}}{\Leftrightarrow} & -y_{j\bar{\ell}\bar{\ell}t} &\leq y_{j\ell\bar{\ell}t}, &&
\end{aligned}
$$

which is true, since the $y$ variables are strictly non-negative. Finally, to show that constraints (15) are also satisfied, it is sufficient to observe that at each location $j$ and capacity level $\ell$, the total flow in the flow conservation constraints (39) and (40) is limited to 1 throughout all time periods. If, for any $j$ and $\ell$, we tried to construct more than 1 facility, this would violate constraints (41) and is thus not possible. Therefore, constraints (15) are satisfied.

If the SIs are used, they are also feasible in the CR-1I model. They can be deduced by replacing (CR.R4) in the SIs of the CR-GMC.

### (B) Construction of a feasible $\overline{\text{CR-GMC}}$ solution from any $\overline{\text{CR-1I}}$ solution

Consider any solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CR-1I}}$. We now construct an equivalent solution $\{x_{ij\ell t}, y_{j\ell_1\ell_2 t}\}$ that is feasible in $\overline{\text{CR-GMC}}$ and has the same objective function value.

The values for the $x_{ij\ell t}$ variables are set identical to those in the CR-1I solution. The arcs for constructing a facility ($y_{j0\ell t}$), closing an open facility ($y_{j\ell\bar{\ell}t}$) and reopening a closed facility ($y_{j\bar{\ell}\ell t}$) are set by using the equalities (CR.R1) - (CR.R3) and therefore satisfy their domain constraints. The solution values for the arcs to keep a facility open ($y_{j\ell\ell t}$) are set by replacing (CR.R1) and (CR.R2) in equality (CR.R4):

$$
\begin{aligned}
& & y_{j\ell t} &= y_{j0\ell t} + y_{j\ell\ell t} + y_{j\bar{\ell}kt} && \text{(CR.R4)}\\
&\overset{(CR.R1),(CR.R2)}{\Leftrightarrow} & y_{j\ell t} &= s_{j\ell t} + y_{j\ell\ell t} + v^o_{j\ell t} && \\
&\Leftrightarrow & y_{j\ell\ell t} &= y_{j\ell t} - s_{j\ell t} - v^o_{j\ell t}. &&
\end{aligned}
$$

The variables are non-negative, as can be verified in equalities (13). Also, due to (41), their values never exceed 1. Constraints (37) and (38) are satisfied, as they contain the same variables with the same values in both models. As shown above in part (A), we can transform equalities (13) into equalities (39), and vice-versa, by using (CR.R1)-(CR.R2). This proves the feasibility of constraints (39). Finally, we compute the values for $y_{j\bar{\ell}\bar{\ell}t}$ by using equalities (40), sequentially from time period 1 to $|T|$:

$$
\begin{aligned}
& & y_{j\bar{\ell}\bar{\ell}t} &= y_{j\bar{\ell}\bar{\ell}(t-1)} + y_{j\ell\bar{\ell}(t-1)} - y_{j\bar{\ell}\ell t} && (40)\\
&\overset{(CR.R2),(CR.R3)}{\Leftrightarrow} & y_{j\bar{\ell}\bar{\ell}t} &= y_{j\bar{\ell}\bar{\ell}(t-1)} + v^c_{j\ell(t-1)} - v^o_{j\ell t}. &&
\end{aligned}
$$

Note that, due to (14), the variables have non-negative values. Furthermore, their sum never exceeds 1, because the only way how to insert flow into the $v^o$ and $v^c$ variables is by using the $s$ variables, whose total sum is strictly limited to 1 by inequalities (15).

We note that the constructed solution has the same value, as can be verified by the used relationships (CR.R1)-(CR.R4) as well as the way the variables' coefficients are composed (see Section 3.2).

From the two parts (A) and (B) above, it follows that $v(\overline{CR - GMC}) = v(\overline{CR - 1I})$. ∎

### A.1.2 CR-GMC and CR-1I are stronger than CR-2I

We next prove that the CR-GMC and CR-1I formulations provide stronger LP bounds than the CR-2I formulation.

**Theorem A.2** $v(\overline{CR\text{-}1I}) \geq v(\overline{CR\text{-}2I})$.

**Proof.** The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{CR-2I}}$ from any feasible solution of $\overline{\text{CR-1I}}$ and that both solutions have the same objective function value. Then, we identify a small example instance where the $\overline{\text{ER-1I}}$ formulation provides a better LP relaxation bound than the $\overline{\text{ER-2I}}$ formulation.

#### (A) Construction of a feasible $\overline{\textbf{CR-2I}}$ solution from any feasible $\overline{\textbf{CR-1I}}$ solution

We now set the solution values for the $s_{j\ell t_1 t_2}$ and $y_{j\ell t_1 t_2}$ variables. For each $j$ and $\ell$, we consider the diagram that describes the opening schedule of a facility of size $\ell$ in the CR-1I solution. We separate the opening schedules for each capacity level $\ell$ into blocks, as described by the following algorithm:

**Algorithm 1.**
**Input**: A facility opening schedule, consisting of a value between 0 and 1, indicating the fraction at which the facility is open for each of the $|T|$ time periods (indicated by the value of $y_{j\ell t}$).
**Output**: The opening schedule horizontally cut into blocks. Each block is defined by a starting and ending period as well as a value between 0 and 1, indicating the fraction at which the block represents the open facility.
**Description**: The opening schedule, as shown in Figure 1 (a), is horizontally cut into blocks whenever the value of the opening fraction increases or decreases. Doing this, the increase and/or decrease of capacity may be split into several increases and/or decreases, respectively. This results in a representation as in Figure 1 (b). In this example, the capacity increase at the beginning of period 3 is split into two capacity increases of half size each, while the capacity decrease at the beginning of period 6 is split into two capacity decreases. It is easy to see that this kind of division is unambiguous, i.e. there is only one way to separate into blocks. The design of an algorithm to find this division is straightforward. We therefore do not explicitly state such an algorithm.

Note that, in the opening schedule, a capacity increase at time period $t$ is always caused by the use of the variables $s_{j\ell t}$ or $v^o_{j\ell t}$. A capacity decrease at time period $t$ is caused by the use of variable $v^c_{j\ell t}$.

After division, we have a number of separated blocks (each spanning over one or more time periods). We divide these blocks into two groups: blocks where the capacity increase is originated from a variable $s_{j\ell t}$ and blocks where the capacity increase is originated from reopening variables $v^o_{j\ell t}$. Each block originated from a variable $s_{j\ell t}$ represents a $s_{j\ell t_1 t_2}$ variable and each block originated from a $v^o_{j\ell t}$ variable represents a variable $y_{j\ell t_1 t_2}$. The value for these variables is set equivalent to the fraction of value represented by the corresponding variables $s_{j\ell t}$ and $v^o_{j\ell t}$.

The following relationship then holds, since the solution value of $y_{j\ell t}$ is the sum of all capacity blocks at time period $t$:

$$\sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} s_{j\ell t_1 t_2} + y_{j\ell t_1 t_2} = y_{j\ell t} \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T. \qquad \text{(CR.R5)}$$

Furthermore, the following relationships hold, since the solution value of $s_{j\ell t}$ is distributed over all $s_{j\ell t_1 t_2}$ variables that initiate at $t_1 = t$. The same relation is valid between the variables $v^o_{j\ell t}$ and $y_{j\ell t_1 t_2}$:

$$\sum_{t_2=t}^{|T|} s_{iktt_2} = s_{j\ell t} \quad \forall j \in J, \ \ \forall \ell \in L, \ \ \forall t \in T. \tag{CR.R6}$$

All $x_{ij\ell t}$ variables are set as given in the CR-1I solution. Thus, constraints (11) are satisfied. Equalities (14) guarantee that, at any time period $t$, variable $v^o_{j\ell t}$ does not hold more capacity than has been previously constructed. Thus, constraints (20) also hold. Using (CR.R6) in (15) shows that constraints (21) are satisfied. Inequalities (22) are also satisfied. To show this, first replace the terms by (CR.R5). Then, recognize that (15) limits the entire facility construction to 1. As $y_{j\ell t}$ is linked to the facility construction in equalities (13), its sum over all capacity levels can never exceed 1. Finally, the capacity constraints (23) are feasible. This is shown by replacing (CR.R5) in constraints (23), which then equal the capacity constraints of the CR-1I. If SIs are used, the feasibility of the SIs in the CR-2I formulation can be shown by replacing its RHS terms by (CR.R5).

We note that the constructed solution has the same value as the CR-1I solution. This can be seen by recognizing that the $s_{j\ell t_1 t_2}$ and $y_{j\ell t_1 t_2}$ blocks in the CR-2I solution have been constructed following the corresponding solution values of $s_{j\ell t}$ and $y_{j\ell t}$ and the way how the cost coefficients are set as described in Section 4.1.2.

#### (B) Problem instance where CR-1I is stronger

Consider the following example instance. We consider a planning over three time periods. A single customer exists with demands of 15, 15 and 20 units for each of the time periods, respectively. Two locations can be used to construct facilities. A single capacity level is available, providing a capacity of 10 units. Construction costs are 100$. Facility closing and reopening is free. The same holds for the production and transportation of the commodity. For the given instance, the CR-1I provides a better bound than the CR-2I formulation: the cost of the solution for the CR-1I model is 2700, whereas the cost of the solution for the CR-2I model is 2650.

From the two parts (A) and (B) above follows that $v(\overline{\text{CR-1I}}) \geq v(\overline{\text{CR-2I}})$. ∎

**Theorem A.3** $v(\overline{\textit{CR-GMC}}) \geq v(\overline{\textit{CR-2I}})$.

**Proof.** The result follows by transitivity from Theorems A.1 and A.2. ∎

### A.1.3 CR-2I+ is equally strong as CR-GMC and CR-1I

**Theorem A.4** $v(\overline{\textit{CR-1I}}) = v(\overline{\textit{CR-2I+}})$.

**Proof.** It has already been shown that we can construct an equivalent and feasible CR-2I solution from any CR-1I solution. Due to the way the described algorithm attributes the values to the $s_{j\ell t_1 t_2}$ and $y_{j\ell t_1 t_2}$ variables as well as the direct relationship between these variables and the $v_c$ and $v_o$ variables, it can be shown that the new constraints (26) are also satisfied.

#### (A) Construction of a feasible $\overline{\text{CR-1I}}$ solution from any feasible $\overline{\text{CR-2I+}}$ solution

Consider any solution $\{x_{ij\ell t}, s_{j\ell_1 t_1 t_2}, y_{j\ell_1 t_1 t_2}\}$ that is feasible in $\overline{\text{CR-2I+}}$. We now construct an equivalent solution $\left\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, v^o_{j\ell t}, v^c_{j\ell t}\right\}$ that is feasible in $\overline{\text{CR-1I}}$ and that has the same value.

We set the values for the $x_{ij\ell t}$ variables identical to those in the CR-2I+ solution. The values for the variables $y_{j\ell t}$, $s_{j\ell t}$, $v^o_{j\ell t}$ and $v^c_{j\ell t}$ are set by establishing the following relations $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$s_{j\ell t} = \sum_{t_2=t}^{|T|} s_{j\ell t t_2} \tag{CR.R7}$$

$$y_{j\ell t} = \sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} s_{j\ell t_1 t_2} + \sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} y_{j\ell t_1 t_2} \tag{CR.R8}$$

$$v^o_{j\ell t} = \sum_{t_2=t}^{|T|} y_{j\ell t t_2} \tag{CR.R9}$$

$$v^c_{j\ell t} = \sum_{t_1=1}^{t-1} s_{j\ell t_1 (t-1)} + \sum_{t_1=1}^{t-1} y_{j\ell t_1 (t-1)} \tag{CR.R10a}$$

$$\Leftrightarrow v^c_{j\ell(t+1)} = \sum_{t_1=1}^{t} s_{j\ell t_1 t} + \sum_{t_1=1}^{t} y_{j\ell t_1 t} \tag{CR.R10}$$

Constraints (11) are equivalent to constraints (19) and are thus satisfied. By using (CR.R8), constraints (12) correspond to constraints (23). Constraints (15) correspond to constraints (22) by using (CR.R7).

Replacing (CR.R9) and (CR.R10) in constraints (14) gives, $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$\sum_{t'=1}^{t} v^o_{j\ell t'} \le \sum_{t'=1}^{t} v^c_{j\ell t'} \tag{14}$$

$$\Leftrightarrow \sum_{t'=1}^{t} v^o_{j\ell t'} \le \sum_{t'=0}^{t-1} v^c_{j\ell(t'+1)}$$

$$\overset{(CR.R9)\&(CR.R10)}{\Leftrightarrow} \sum_{t'=1}^{t}\sum_{t_2=t'}^{|T|} y_{j\ell t' t_2} \le \sum_{t'=0}^{t-1}\sum_{t_1=1}^{t'} s_{j\ell t_1 t'} + \sum_{t'=0}^{t-1}\sum_{t_1=1}^{t'} y_{j\ell t_1 t'}$$

$$\Leftrightarrow \sum_{t'=1}^{t}\sum_{t_2=t'}^{|T|} y_{j\ell t' t_2} \le \sum_{t_1=1}^{t-1}\sum_{t'=t_1}^{t-1} s_{j\ell t_1 t'} + \sum_{t_1=1}^{t-1}\sum_{t'=t_1}^{t-1} y_{j\ell t_1 t'},$$

which is true due to constraints (26). Thus, constraints (14) also hold. The feasibility of the flow conservation constraints (13) can be shown by replacing the variables by the terms given in the relations (CR.R7), (CR.R9) and (CR.R10a). By doing so, all terms on the LHS and RHS will cancel each other.

Given the relations (CR.R7)-(CR.10a) and the way the variable coefficients are composed in both formulations, it can easily be verified that both solutions have the same value. Both formulations are thus equally strong. ∎

**Theorem A.5** $v(\overline{CR\text{-}GMC}) = v(\overline{CR\text{-}2I+})$.

**Proof.** The result follows by transitivity from Theorems A.1 and A.4. ∎

## A.2 Theoretical Results for the DMCFLP_ER formulations

We now prove the dominance relationships for the three formulations presented for the DMCFLP_ER. Let $\overline{\text{ER-GMC}}$ be the linear programming relaxation of ER-GMC. In the same way, we denote $\overline{\text{ER-1I}}$ the linear programming relaxation of ER-1I and $\overline{\text{ER-2I}}$ the linear programming relaxation of ER-2I.

### A.2.1 ER-1I and ER-2I are equally strong

We next prove that the LP relaxations of the formulations ER-1I and ER-2I provide the same lower bound.

**Theorem A.6** $v(\overline{ER\text{-}1I}) = v(\overline{ER\text{-}2I})$.

**Proof.** The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{ER-1I}}$ from any feasible solution of $\overline{\text{ER-2I}}$ and that both solutions have the same objective function value. Then, we show the same, constructing an equivalent $\overline{\text{ER-2I}}$ solution based on any feasible $\overline{\text{ER-1I}}$ solution.

### (A) Construction of a feasible $\overline{\text{ER-1I}}$ solution from any $\overline{\text{ER-2I}}$ solution

Consider any solution $\{x_{ij\ell t}, y_{j\ell t_1 t_2}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\}$ that is feasible in $\overline{\text{ER-2I}}$. We now construct an equivalent solution $\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\}$ that is feasible in $\overline{\text{ER-1I}}$ and has the same objective function value.

We set all variables $x_{ij\ell t}, y_{j\ell t}$, $s_{j\ell t}$ and $w_{j\ell t}$ in the $\overline{\text{ER-1I}}$ formulation as given in the $\overline{\text{ER-2I}}$ solution. Given that both formulations have the same objective function, the solution value is also the same. Also observe that the formulations have the same constraints except for constraints (28) in the ER-1I and constraints (32) - (34) in the ER-2I formulation. The constraints that are part of both models (including the SIs) have the same variables in both models with the same solution values in both solutions and are thus feasible. Therefore, we only have to show that constraints (28) are also feasible. We do so by replacing (32) - (34) in (28) for $\forall j \in J, \forall t \in T$:

$$\sum_{\ell\in L} \ell y_{j\ell t} = \sum_{\ell\in L} \ell y_{j\ell(t-1)} + \sum_{\ell\in L} \ell s_{j\ell t} - \sum_{\ell\in L} \ell w_{j\ell t} \tag{28}$$

$$\overset{(32)-(34)}{\Leftrightarrow} \sum_{\ell\in L}\sum_{t_1=1}^{t}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t_1 t_2} = \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\sum_{t_2=t-1}^{|T|} \ell y'_{j\ell t_1 t_2} + \sum_{\ell\in L}\sum_{t_2=t}^{|T|} \ell y'_{i\ell(t_1=t)t_2} - \sum_{\ell\in L}\sum_{t_1=1}^{t-1} \ell y'_{i\ell t_1(t_2=t-1)}$$

$$\Leftrightarrow \quad \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\sum_{t_2=t}^{|T|} \ell y'_{j\ell t_1 t_2} + \sum_{\ell\in L}\sum_{t_2=t}^{|T|} \ell y'_{j\ell(t_1=t)t_2}$$

$$= \sum_{\ell\in L}\sum_{t_1=1}^{t-1}\sum_{t_2=t-1}^{|T|} \ell y'_{j\ell t_1 t_2} + \sum_{\ell\in L}\sum_{t_2=t}^{|T|} \ell y'_{i\ell(t_1=t)t_2} - \sum_{\ell\in L}\sum_{t_1=1}^{t-1} \ell y'_{i\ell t_1(t_2=t-1)}.$$

The remaining terms now cancel each other and therefore constraints (28) hold.

### (B) Construction of a feasible $\overline{\text{ER-2I}}$ solution from any $\overline{\text{ER-1I}}$ solution

Consider any solution $\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\}$ that is feasible in $\overline{\text{ER-2I}}$. We now construct an equivalent solution $\{x_{ij\ell t}, y_{j\ell t_1 t_2}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\}$ that is feasible in $\overline{\text{ER-1I}}$ and has the same objective function value.

We set the values for the $x_{ij\ell t}, y_{j\ell t}$, $s_{j\ell t}$ and $w_{j\ell t}$ variables in the $\overline{\text{ER-2I}}$ formulation as given in the $\overline{\text{ER-1I}}$ solution. Given that both formulations have the same objective function, the solution value in the objective function is also the same. All constraints (including the SIs), except for constraints (32) - (34) are the same as in formulation ER-1I and are therefore feasible.

We now set the solution values for the $y_{j\ell t_1 t_2}$ variables. For each $\ell \in L$, we consider the diagram that describes the opening schedule of a facility of size $\ell$. Each opening schedule is horizontally cut into blocks as described by Algorithm 1. Note that in the optimal solution, due to (28), an increase in $y_{j\ell t}$ by an amount of $\alpha$ necessarily means that $s_{j\ell t} = \alpha$ and $w_{j\ell t} = 0$, whereas a decrease in $y_{j\ell t}$ by an amount of $\alpha$ necessarily means that $w_{j\ell t} = \alpha$ and $s_{j\ell t} = 0$. After separation, each of the separated blocks represents a variable $y_{j\ell t_1 t_2}$ with a solution value greater than 0. The solution value of $s_{j\ell t}$ will be distributed over all $y_{j\ell t_1 t_2}$ variables that start at $t_1 = t$, the solution value of $w_{j\ell t}$ will be distributed over all $y_{j\ell t_1 t_2}$ variables that terminate at the end of $t_2 = t$ and the solution value of $y_{j\ell t}$ will be distributed over all $y_{j\ell t_1 t_2}$ variables that start at or before $t$ and terminate at or after $t$. Therefore, the following relationships hold:

$$\sum_{t_2=t}^{|T|} y'_{i\ell t t_2} = s_{j\ell t} \quad \forall j \in J, \;\; \forall \ell \in L, \;\; \forall t \in T$$

$$\sum_{t_1=1}^{t-1} y'_{i\ell t_1(t-1)} = w_{j\ell t} \quad \forall j \in J, \;\; \forall \ell \in L, \;\; \forall t \in T$$

$$\sum_{t_1=1}^{t} \sum_{t_2=t}^{|T|} y'_{j\ell t_1 t_2} = y_{j\ell t} \quad \forall j \in J, \;\; \forall \ell \in L, \;\; \forall t \in T.$$

Replacing these relationships in the constraints (32) - (34), respectively, shows that these constraints remain feasible.

From the two parts (A) and (B) above follows that $v(\overline{ER-1I}) = v(\overline{ER-2I})$. ∎

### A.2.2 ER-GMC is stronger than ER-1I and ER-2I

We now compare the strength of the ER-GMC and ER-1I formulations. We will prove that the ER-GMC formulation is at least as strong (strictly stronger for some instances) as the ER-1I formulation in the sense that its linear programming relaxations provides a better bound. By transitivity, the same result follows for the relation between the ER-GMC and ER-2I.

**Theorem A.7** $v(\overline{ER\text{-}GMC}) \geq v(\overline{ER\text{-}1I})$.

**Proof.** The proof consists of two parts: First, we show how to construct a feasible solution for $\overline{\text{ER-1I}}$ from any feasible solution of $\overline{\text{ER-GMC}}$ and that both solutions have the same objective function value. Second, we provide a problem instance where $\overline{\text{ER-GMC}}$ provides a better bound than $\overline{\text{ER-1I}}$.

#### (A) Construction of a feasible $\overline{\text{ER-1I}}$ solution from any $\overline{\text{ER-GMC}}$ solution

Consider any solution $\{x_{ij\ell t}, y_{j\ell_1\ell_2 t}\}$ that is feasible in $\overline{\text{ER-GMC}}$. We now construct an equivalent solution $\{x_{ij\ell t}, y_{j\ell t}, s_{j\ell t}, w_{j\ell t}\}$ that is feasible in $\overline{\text{ER-1I}}$ and has the same objective function value.

We deduce the values for the new variables from the ones of the existing solution variables $y_{j\ell_1\ell_2 t}$. Equalities (4) in the ER-GMC formulation conserve the flow for open facilities as it is found at the end of each planning period. It can be used to deduce the values for the $y$ variables $\forall j \in J, \forall \ell \in L, \forall t \in T$:

$$y_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1 \ell t}. \tag{ER.R1}$$

The same equalities (4) also lead to the following result:

$$y_{j\ell(t-1)} \overset{(ER.R1)}{=} \sum_{\ell_1 \in L} y_{j\ell_1\ell(t-1)} \overset{(4)}{=} \sum_{\ell_2 \in L} y_{j\ell\ell_2 t}. \tag{ER.R2}$$

Furthermore, we set $s_{j\ell t}$ and $w_{j\ell t}$ as follows:

$$s_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1(\ell_2=\ell_1+\ell)t} \tag{ER.R3}$$

$$w_{j\ell t} = \sum_{\ell_1 \in L} y_{j\ell_1(\ell_2=\ell_1-\ell)t}. \tag{ER.R4}$$

Having set the variables for the ER-1I formulation, we now show that the equalities (28) still hold. We replace the variables by the deduced values according to (ER.R1)-(ER.R4):

$$\sum_{\ell \in L} \ell y_{j\ell t} + \sum_{\ell \in L} \ell w_{j\ell t} = \sum_{\ell \in L} \ell y_{j\ell(t-1)} + \sum_{\ell \in L} \ell s_{j\ell t} \quad \forall j \in J, \quad \forall t \in T \tag{28}$$

$$\overset{(ER.R1)-(ER.R4)}{\Leftrightarrow} \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\ell_2 t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_2=\ell_1-\ell)t}$$

$$= \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_1 y_{j\ell_1\ell_2 t} + \sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_2=\ell_1+\ell)t} \quad \forall j \in J, \quad \forall t \in T. \tag{44}$$

In the following, we prove that (44) is true by using the principle of induction:

**Proposition**: Equalities (44) are true for all sizes of $L$.

**Basic cases**: We start with the trivial case of $q = 1$, i.e., $L = \{0, 1\}$. Note that, for the sake of simplicity, we suppress the variable indices $j$ and $t$, but indicate only the values for the indices $\ell_1$ and $\ell_2$:

$$LHS : \sum_{\ell_1 \in L} \sum_{\ell_2 \in L} \ell_2 y_{j\ell_1\ell_2 t} \rightarrow 0y_{00} + 1y_{01} + 0y_{10} + 1y_{11}$$

$$\sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_2=\ell_1-\ell)t} \rightarrow 1y_{10}$$

$$RHS : \sum_{\ell_1 \in L} \sum_{\ell_1 \in L} \ell_1 y_{j\ell_1 k t} \rightarrow 0y_{00} + 0y_{01} + 1y_{10} + 1y_{11}$$

$$\sum_{\ell_1 \in L} \sum_{\ell \in L} \ell y_{j\ell_1(\ell_2=\ell_1+\ell)t} \rightarrow 1y_{01}$$

It can be easily verified that the terms on the LHS equal the terms on the RHS. The proposition is thus true for $q = 1$.

**Inductive step**: We now show that the proposition also holds for $q = q+1$. For $q+1$, the LHS and RHS include the same terms as in the previous step. In addition, the following terms are added:

$$LHS : \sum_{\ell_2=0}^{q+1} \ell_2 y_{j(\ell_1=q+1)\ell_2 t} + \sum_{\ell_1=0}^{q} (q+1) y_{j\ell_1(\ell_2=q+1)t}$$

$$\sum_{\ell_2=0}^{q+1} (q+1-\ell_2) y_{j(\ell_1=q+1)\ell_2 t}$$

$$RHS : \sum_{\ell_2=0}^{q+1} (q+1) y_{j(\ell_1=q+1)\ell_2 t} + \sum_{\ell_1=0}^{q} \ell_1 y_{j\ell_1(\ell_2=q+1)t}$$

$$\sum_{\ell_1=0}^{q+1} (q+1-\ell_1) y_{j\ell_1(\ell_2=q+1)t}$$

Summing up all terms on the LHS and all terms on the RHS shows that both sides are equivalent. Hence the result follows by induction.

Therefore, constraints (28) are satisfied. For the $x_{ij\ell t}$ variables, we choose the same solution values as in the ER-GMC solution. Constraints (11) are therefore necessarily satisfied. In addition, the demand allocation contributes equally to the objective function in both formulations. Constraints (12) are also satisfied, as can be verified by replacing the $y_{j\ell_1\ell_2 t}$ variables in constraints (3) by (ER.R1). The limiting constraints (see

Section 4.2) are also satisfied by noting that each of the variables can be replaced by corresponding $y_{j\ell_1\ell_2 t}$ variables and the sum of all $y_{j\ell_1\ell_2 t}$ variables never exceeds 1. Finally, the SIs are feasible due to relationship (ER.R1).

The contribution of the variables $y_{j\ell t}$, $s_{j\ell t}$ and $w_{j\ell t}$ to the total solution costs is equivalent to the one of the $y_{j\ell_1\ell_2 t}$ variables. This can be easily shown by verifying the equalities (ER.R1)-(ER.R4) and the costs attributed to the $y_{j\ell_1\ell_2 t}$ in Section 3.2.

**(B) Problem instance where ER-GMC is stronger**

We now explain, by the use of a small problem instance, under which circumstances the ER-GMC provides a better LP bound than the ER-1I and ER-2I formulations.

The example instance we explore here contains one potential facility location and one client. The planning horizon contains one single-time period. The client has a demand of 10 units in time period 1 and 2 units in time period 2. Production and transportation of the demanded commodities is free. The maximum capacity level is 2. The capacity expansion costs 200$ for one capacity level and 350$ for two capacity levels. Capacity reduction costs 20$ for one capacity level and 35$ for two capacity levels. Maintenance costs for a facility is 300$ at capacity level 1 and 500$ at capacity level 2. The facility capacity is 10 at level 1 and 11 at level 2. Therefore, the costs to provide and maintain a certain amount of capacity do not follow the principle of economies of scale.

The ER-GMC formulation provides a better bound than the other formulations. Consider the solution we found before for the ER-GMC formulation: $y_{j_0\ell_0\ell_1 t_0} = 1.0$ and $x_{i_0 j_0 \ell_2 t_0} = 1.0$. The solution value is 500. Now, adding the SIs, we $y_{j_0\ell_0\ell_1 t_0} = 1.0$ and $x_{i_0 j_0 \ell_1 t_0} = 1.0$. For the ER-1I and ER-2I formulations, with our without the SIs, we still have the possibility to construct half a level 2 facility, while maintaining a full level 1 facility. That is, the decision variables linked to the objective function have the solution values: $y_{j_0\ell_1 t_0} = 1.0$ and $s_{j_0\ell_2 t_0} = 0.5$. The objective function value is then 475.

From the two parts (A) and (B) above, it follows: $v(\overline{\text{ER-GMC}}) \geq v(\overline{\text{ER-1I}})$. ∎

**Theorem A.8** $v(\overline{\textit{ER-GMC}}) \geq v(\overline{\textit{M-ER-I2}})$.

**Proof.** The result follows by transitivity from Theorems A.6 and A.7. ∎