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Service Level Robustness in Stochastic Production Planning under Random Machine Breakdowns

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Abstract In this paper, we consider a multi-period, multi-product production planning problem where the production rate and the customer service level are random variables due to machine breakdowns. In order to determine robust production plans, constraints are introduced in the stochastic capacitated lot-sizing problem to ensure that a pre-specified customer service level is met with high probability. The probability of meeting a service level is evaluated by using the first passage time theory of a Wiener process to a boundary. A two-step optimization approach is proposed to solve the developed model. In the first step, the mean-value deterministic model is solved. Then, a method is proposed in the second step to improve the probability of meeting service level. The resulting approach has the advantage of not being a scenario-based one. It is shown that substantial improvements in service level robustness are often possible with minimal increases in expected cost.

Keywords. Robust production planning, random failures, service level, first passage time, Brownian motion.

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1. Introduction

We are concerned with the problem of ensuring robust customer service levels when addressing a multi-period, multi-product (MPMP) production planning problem in a manufacturing system subject to random failures and repairs. The service level is measured by the fraction of demand satisfied on time, and due to random failures, it is a stochastic process. In this context, it is important to evaluate the probability of meeting such a service level. Higher probability corresponds to more robust service level. In fact, if a production manager promises to a customer the satisfaction of his demand with a given service level, it is important to honour her/his promise. Otherwise, not only some of the required demand will not be delivered, but even the promised (reduced) quantities will not be delivered on time. Measuring a service level and its robustness are directly motivated from manufacturing and distribution practices and represent decisive topics to investigate (Hausman et al., 1998). Even if there is a substantial literature on the mathematical analysis of manufacturing systems, few papers are concerned with the service level robustness. A literature survey on performance evaluation of manufacturing systems can be found for example in (Dallery and Gershwin, 1992). The goal of a large number of the existing work is to predict the production rate and average inventory levels. In (Chew and Johnson, 1996), the authors develop an approximate method for predicting the service level of a multi-echelon inventory system. There have been also some studies in the literature on explicit evaluation of the output variability. In (Tan, 1999), a discrete flow model is used to estimate the variance of the production output as a function of time. In (Ciprut et al., 1999) and (Nourelfath and Hongler, 2004), this variance is evaluated by using a fluid modelling approach. The later consists in using continuous variables to characterize the flow of parts, and obviously involves continuous stochastic processes. In this paper, we adopt also a fluid modelling approach to derive a closed form expression of the probability of meeting a service level. More precisely, we consider the problem of determining quantities of items (lot sizes) to be produced during each period of the planning horizon. The objective in lot-sizing models is to minimize the total cost in the planning horizon, regarding fulfilment of products demand, machines capacities, etc. Solution methodologies vary from traditional linear mixed integer programming, and associated branch and bound exact methods to heuristic methods: see for example (Wolsey, 2002) for a survey. The setting of lot sizes is usually considered as a decision related to tactical planning, which is a medium-term activity. In aggregate planning, the lot sizing models are extended by including labor resource decisions (Shapiro, 1993; Sipper, 1997; Wolsey, 2002; Wienstein and Chung, 1999). Tactical planning bridges the transition from the strategic planning level (long-term) to the operational planning level (short-term). At all production planning levels, it is very important to take into account both system uncertainties (machine failures) and environmental uncertainties such as products demands and processes yields. In spite of this, traditional mathematical programming models for production planning tend to be deterministic (Shapiro, 1993; Sipper, 1997), leading to unsatisfactory production plans in the presence of such uncertainties.

A review of some of the existing literature on production planning under uncertainty is provided in (Mula et al., 2006). There are two bodies of literature that are related to the present paper: production planning models with service level constraints using stochastic and robust optimization tools; and reliability models. We briefly review some important papers in each category.

Silver (1978) proposed a heuristic procedure to solve the stochastic dynamic single-item uncapacitated lot-sizing problem considering different service level measures. This heuristic is based on a simple, but effective, procedure for the deterministic case. Askin (1981) proposed a heuristic that solves the same problem but which includes the cost effects of the demand randomness. This heuristic uses an order-up-to policy combined with a least period cost approach. Tarim and Kingsman (2004) addressed the multi-period single-item inventory lot-sizing problem with stochastic demands under the ‘‘static–dynamic uncertainty’’ strategy in (Bookbinder and Tan, 1988). In this strategy, the replenishment periods are fixed at the beginning of the planning horizon, but the actual orders are determined only at those replenishment periods and depend upon the demand that is realised. In a follow-up paper, Tarim and Kingsman (2006) considered backorder costs instead of service level. In (Tempelmeier, 2007), the author considered the uncapacitated single-item dynamic lot-sizing problem with stochastic period demands and backordering. He presented a model formulation that minimizes the setup and holding costs with respect to service level constraints. In (Vargas, 1999), it was shown that for a model formulation with backorder costs and stationary unit production costs the exact solution can be found by using a shortest-path algorithm. For the same model but with non-stationary unit production costs, Sox (1997) formulated a nonlinear mixed-integer program and developed a dynamic programming algorithm. Within this context, stochastic programming (Dantzig, 1955; Kall and Wallace, 1994; Birge and Louveaux, 1997; Kall and Mayer, 2005) and robust optimisation (Mulvey et al., 1995) have seen several successful applications in production planning. In (Escudero et al., 1993) a multi-stage stochastic programming approach was used for addressing an MPMP production planning model with random demand. Alfieri and Brandimarte (2005) reviewed the multi-stage stochastic models applied in multi-period production and capacity planning in the manufacturing systems. Leung and Wu (2004) proposed a robust optimisation model for stochastic aggregate production planning. Wu (2006) applied the robust optimisation approach to uncertain production loading problems with import quota limits under the global supply chain management environment. In (Leung et al., 2007) a robust optimisation model was developed to address a multi-site aggregate production planning problem in an uncertain environment. Kazemi Zanjani et al. (2009) proposed a multi-stage stochastic programming approach for multi-product capacitated lot-sizing with uncertain demand and random processes yields. Kazemi Zanjani et al. (2010) proposed two robust optimisation models with different recourse cost variability measures to address MPMP production planning with uncertain yield.

In reliability, there exist also many papers dealing with production planning under machine failures (Cho and Parlar, 1991; Dekker, 1996; Marseguerra et al., 2005). Generally, the objective of these planning models is either to maximize the availability, or to minimize the maintenance cost. These models are generally solved by coupling optimization methods with analytical tools or simulation. In (Panda et al., 2008), the authors have considered an economic production lot-size model for imperfect products in which production rate is a fixed quantity and the demand rate is probabilistic. Lee (2008) has developed a maintenance model in multi-level multi-stage system. This model gives the optimal amount of investment in preventive maintenance that reduces the variance from the target value of the quality characteristics. In (Sana, 2010), the author has investigated an economic production lot-size model in an imperfect production system. The total costs in this investment model include manufacturing cost, setup cost, holding cost and reworking cost of imperfect quality products. The author shows that the production cost per unit item is convex function of production rate.

In the above-mentioned papers, machine breakdowns are either ignored, or studied from a reliability optimization perspective. While it is vital for production managers to take into account machine failures, such a problem is regretfully complex when viewed with a production planning optimization perspective. It is indeed possible to use stochastic and robust optimization approaches to model random failures and repairs. However, these approaches are scenarios-based and they may lead to models of non manageable sizes. In fact, their main drawback is the complexity of the resulting optimization models that become computationally intractable as the number of scenarios increases. In a recent contribution applying robust optimization approaches in sawmill production planning (Kazemi Zanjani et al., 2010), we have shown that there is a trade-off between the service level robustness and the expected cost. In a very service sensitive company that wants to establish a reputation for always meeting customer service level, the robust optimization formulation in (Kazemi Zanjani et al., 2010) allows a decision maker to see explicitly what possible trade-offs between service level variability and the expected cost exists, and to choose a solution that is consistent with his/her willingness to accept risk. In this paper, we adopt a conceptually different point of view to represent the trade-off between the service level robustness and the expected cost increase, when considering random machine failures and repairs. The importance of our contribution lies in the fact that, unlike classical stochastic optimization and robust optimization methods, the proposed approach is *not* a scenario-based one. It incorporates, with a mean value deterministic model, an improvement method of the service level robustness using the first passage time theory of a Wiener process to a boundary.

The remainder of the paper is organized as follows. Section 2-4 presents, respectively, the proposed model, the solution method and an illustrative example. Concluding remarks are in Section 5.

2. The model

2.1. *Failure-prone manufacturing system*

Consider a manufacturing system producing a set of products P during a given planning horizon H including T periods. Each period t (t = 1, 2, …, T) has a fixed length T_t. For each product $p \in P$, a customer demand d_{pt} is to be satisfied at the end of period t. The system is subjected to random machine breakdowns, and it can perform its task with two levels of performance: states 0 and 1. State 1 is a perfect functioning state where the system has a production rate g. State 0 is a complete failure state where the system is not producing. The probability distribution of the time between failures is denoted as $F(t)$ and the probability distribution of the time needed to repair as $H(t)$. We assume that these probability measures have respectively the densities $f(t)$ and $h(t)$ and that their moments are finite. The failure and the repair rates are denoted by λ and μ, respectively. We assume that the parameters of the system (production, failure and repair rates) do not depend on the kind of product. The mean time between failures (MTBF) is:

$$\frac{1}{\lambda}=\int_0^{\infty} t\ dF(t)=\int_0^{\infty} t\ f(t)\ dt\,. \tag{1}$$

The variance and the squared coefficient of variations are, respectively, defined by:

$$\sigma_\lambda^2=\int_0^{\infty} t^2\ f(t)\ dt-\left(\frac{1}{\lambda}\right)^2,\ \ CV_\lambda{}^2=\sigma_\lambda^2\,\lambda^2\,. \tag{2}$$

Similarly, the mean time to repair (MTTR) is:

$$\frac{1}{\mu}=\int_0^{\infty} t\ dH(t)=\int_0^{\infty} t\ h(t)\ dt\,, \tag{3}$$

and the variance and the squared coefficient of variations are:

$$\sigma_\mu^2=\int_0^{\infty} t^2\ g(t)\ dt-\left(\frac{1}{\mu}\right)^2,\ \ CV_\mu{}^2=\sigma_\mu^2\,\mu^2\,. \tag{4}$$

We assume that there exists a steady-state distribution of state probabilities. The steady-state probability of a given state k (k = 0, 1) is denoted by p_k. Because the system is failure-prone, the production rate is a stochastic process, for which the average and the variance are (Ciprut et al., 1999):

$$\langle G \rangle = g\ p_1 = g \frac{\mu}{\lambda+\mu}\ ,\ \sigma_G{}^2 = \ g^2\left(CV_\lambda{}^2 + CV_\mu{}^2\right)\left\{\frac{\lambda\mu}{\left(\lambda+\mu\right)^3}\right\}. \tag{5}$$

The squared coefficient of variation of the production rate is then:

$$CV_G{}^2 = \left(\frac{\sigma_G}{\langle G \rangle}\right)^2 = \frac{\lambda\left(CV_\lambda{}^2 + CV_\mu{}^2\right)}{\mu\left(\lambda+\mu\right)}. \tag{6}$$

2.2. *General description of the problem*

The production planning problem considered in this paper is a stochastic multi-product capacitated lot-sizing problem. The decisions involve determination of quantities of items (lot sizes) to be produced in each period. Lot-sizing is one of the most important problems in tactical production planning. Almost all manufacturing situations involving a product-line contain capacitated lot-sizing problems, especially in the context of batch production systems. The objective function is to minimize the sum of production costs, while satisfying the demand for all products over the entire horizon. The constraints are related to the capacity, the set-up and the dynamics of the inventory and the backorder. The main particularity of the proposed model is that it introduces constraints to ensure that pre-specified target customer service levels are met with high probabilities. This variation of the capacitated lot-sizing problem is modelled to determine the production plans with robust customer service level. Before introducing the mathematical model, let us introduce some definitions related to service level.

2.3. *Definitions*

The backorder size of product p by the end of period t, denoted by B_{pt}, is governed by a stochastic process. We assume that the demand d_{pt} is deterministic. The service level SL_{pt} is a stochastic process that can be defined, for a period t and a product p, by the fraction of demand satisfied on time:

$$SL_{pt} = \frac{d_{pt} - B_{pt}}{d_{pt}}. \tag{7}$$

Let denote by $|P|$ the number of products. We can also define average service levels as follows:

$$SL_t = \sum_{p\in P} \frac{SL_{pt}}{|P|}\ , \tag{8}$$

$$SL_p = \sum_{t=1}^{T} \frac{SL_{pt}}{T}, \tag{9}$$

$$SL = \sum_{p\in P} \sum_{t=1}^{T} \frac{SL_{pt}}{T|P|}. \tag{10}$$

For a period t and a product p, the probability β_{pt} of meeting a service level SL_{pt}^{*} is defined as:

$$\beta_{pt} = Prob\left(SL_{pt} \geq SL_{pt}*\right). \quad (11)$$

Similarly, the following probabilities are defined:

$$\beta_{p} = Prob\left(SL_{p} \geq SL_{p}*\right), \quad (12)$$

$$\beta_{t} = Prob\left(SL_{t} \geq SL_{t}*\right), \quad (13)$$

$$\beta = Prob\left(SL \geq SL*\right). \quad (14)$$

2.4. *Mathematical model*

Minimize
$$\sum_{p\in P}\sum_{t=1}^{T}\left(h_{pt}I_{pt} + b_{pt}B_{pt} + \pi_{pt}x_{pt} + s_{pt}y_{pt}\right), \quad (15)$$

Subject to
$$x_{pt} - I_{pt} + I_{p(t-1)} + B_{pt} - B_{p(t-1)} = d_{pt}, \;\; p\in P, \;\; t=1,\; 2,\; ...,\; T, \quad (16)$$

$$x_{pt} \leq \left(\sum_{q\geq t} d_{pq}\right) y_{pt}, \;\; p\in P, \;\; t=1,\; 2,\; ...,\; T, \quad (17)$$

$$\sum_{p\in P} x_{pt} \leq G\,T_{t}, \;\; t=1,\; 2,\; ...,\; T, \quad (18)$$

$$SL_{pt} \geq SL_{pt}*, \;\; p\in P, \;\; t=1,\; 2,\; ...,\; T, \quad (19)$$

$$\beta_{pt} \geq \beta_{pt}*, \;\; p\in P, \;\; t=1,\; 2,\; ...,\; T, \quad (20)$$

$$x_{pt},\, I_{pt},\, B_{pt} \in \mathbb{N};\; y_{pt} \in \{0,\; 1\}. \quad (21)$$

The objective function (15) consists of:

- a total holding cost of the inventory $\sum_{p\in P}\sum_{t=1}^{T} h_{pt}I_{pt}$, with h_{pt} is the inventory holding cost per unit of product p by the end of period t, and I_{pt} is the inventory level of product p at the end of period t;
- a backorder cost (backlogs are allowed) $\sum_{p\in P}\sum_{t=1}^{T} b_{pt}B_{pt}$, with b_{pt} is the backorder cost (lost opportunity and goodwill) per unit of product p by the end of period t, and I_{pt} is the inventory level of product p at the end of period t;

- a total production cost $\sum_{p\in P}\sum_{t=1}^{T}\pi_{pt}x_{pt}$, with π_p is the variable cost of producing one unit of product p in period t, and x_p is the quantity of product p to be produced in period t; and
- a total setup cost $\sum_{p\in P}\sum_{t=1}^{T}s_{pt}y_{pt}$, with s_{pt} is the fixed set-up cost of producing product p in period t, and y_{pt} is a binary decision variable. Each time that production of item p begins in a period t, a setup must take place and y_{pt} is equal to 1; otherwise, y_{pt} is equal to 0.

The first constraint (16) relates inventory or backorder at the start and end of period t to the production and demand in that period. There is no optimal solution where $I_{pt} > 0$ and $B_{pt} > 0$ simultaneously, since the objective function can be improved by decreasing both I_{pt} and B_{pt} until one becomes zero. Equation (16) ensures simply that the sum of inventory (or backorder) of product p at the end of period t is equal to its inventory (or backorder) in the previous period plus the total production of that product in that period, minus the product demand for that period. For $t = 0$, we assume that $B_{p0} = I_{p0} = 0$ and $x_{p0} = d_{p0} = 0$. The second constraint (17) forces $x_{pt} = 0$ if $y_{pt} = 0$ and frees $x_{pt} \geq 0$ if $y_{pt} = 1$. In equation (17), the quantity $\left(\sum_{q\geq t}d_{pq}\right)$ is an upper bound of x_{pt}. Equation (18) corresponds to the available production capacity constraint.

Equation (19) ensures that each service level SL_{pt} is higher than a pre-specified target value SL_{pt}^{*}. Equation (20) ensures that each probability β_{pt} of meeting the service level SL_{pt}^{*} is higher than a pre-specified target value β_{pt}*.

The model (15)-(21) represents a stochastic optimization problem. As an approximation, the mean-value deterministic model can be formulated by considering the expected values of the random variables. Solving this mean-value equivalent model relies on the evaluation of the probability β_{pt} used in the chance constraints (20). The difficulty of solving the mean-value model with constraints (20) will be discussed after presenting the method used to evaluate the probability β_{pt}.

3. The solution method

3.1. *Evaluation of the probability to meet a service level*

3.1.1. *Basic idea*

The instantaneous cumulative production $x(t)$ represents the number of parts produced during a time interval $[0, t]$. The instantaneous production rate $G(t)$ is linked to $x(t)$ via the differential equation:

$$\frac{d}{dt}x(t) = G(t);\ x(0) = x_0 \ and\ G(0) = g\ . \tag{22}$$

Due to machine breakdowns, $x(t)$ is a stochastic process and Equation (22) is a stochastic differential equation in which the noise source is $G(t)$. For asymptotically large times, we assume that $x(t)$ reaches a stationary regime. This assumption holds since we are concerned with tactical production planning. In fact, even if the time horizon for tactical production planning may vary depending on the industry, a typical value is one month (or more) that is sufficient for the system to reach a stationary regime. Due to the central limit theorem, the cumulative production can be characterized by a Gaussian law. The asymptotic dynamics of $x(t)$ can be described by a drifted Brownian motion (Ciprut et al., 1999). It is well known that the first passage time of such a Wiener process to a boundary has an inverse Gaussian (IG) distribution: see for example (Seshadri, 1993). The probability β_{pt} of meeting a service level is evaluated as a function of this IG distribution. The distribution function of the first passage time of Brownian motion with positive drift was derived by Schrödinger (1915) and Smoluchowski (1915). In two important works, Tweedie (1957a,b) profiled the statistical properties of IG distributions. In (Chen et al., 2004), the authors propose a simulation algorithm to estimate means, variances, and covariances for a set of order statistics from IG distributions. While the works of Schrödinger and Smoluchowski seem to be the earliest references to this law, many answers to questions relating to the physical phenomenon of Brownian motion had already been announced in an ingeniously heuristic way in (Bachelier, 1900). At the basis of such relatively "old" contributions, we evaluate the probability β_{pt} of meeting a target service level.

3.1.2. *A closed form approximation*

Under the assumption of a deterministic demand, using Equations (7) and (11), we have:

$$\beta_{pt} = Prob\left(B_{pt} \leq B_{pt}*\right), \tag{23}$$

where the backorder B_{pt} is a random variable (since the produced quantity x_{pt} is random), and $B_{pt}*$ is a target backorder given by the quantity $B^*_{pt} = d_{pt} - SL^*_{pt}\ d_{pt}$.

From Equation (16), we have $B_{pt} = d_{pt} - x_{pt} + B_{p(t-1)} - I_{p(t-1)}$. It follows that:

$$\beta_{pt} = Prob\left(x_{pt} \geq d_{pt} - B_{pt}*+B_{p(t-1)} - I_{p(t-1)}\right). \tag{24}$$

The operation time needed to produce the quantity x_{pt} is denoted by OT_{pt}. It is also a random variable for which the average is given by:

$$\left\langle OT_{pt}\right\rangle = \frac{\left\langle x_{pt}\right\rangle}{\left\langle G\right\rangle} = \frac{\left\langle x_{pt}\right\rangle}{g\ \mu}\ \left(\lambda+\mu\right), \tag{25}$$

with $\left\langle x_{pt}\right\rangle$ is the average of x_{pt}.

Note that since the quantity x_{pt} is a decision variable, the manufacturing facility may produce during a time that is shorter than the length period T_t (in particular, if it is decided not to produce a product p during a given period t, the operation time OT_{pt} is 0). Because of this, in all the mathematical developments of this paper, the cumulative production characterized during the time interval [0, OT_{pt}] with $0 \leq OT_{pt} \leq T_t$.

Proposition 1. For every period t and every product p, the probability β_{pt} of meeting a service level $SL_{pt}{}^*$ can be approximated by:

$$\beta_{pt}(z) \approx 1 - \frac{1}{2} Erfc\left(\frac{1}{CV_G} \sqrt{\frac{z}{2}} \left(1 - \frac{\langle OT_{pt} \rangle}{z} \right) \right), \qquad (26)$$

where the variable $z \geq \langle OT_{pt} \rangle$ is the effective operation time, and $Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)\, dt$.

Proof. The detailed proof is given in Appendix. It is composed of the following three steps:

Step 1 – The cumulative production $x(t)$ is characterized as a Wiener process, for which the transition probability density is evaluated by a Gaussian law:

$$P(x,t \mid x_0, 0) = \frac{1}{\sqrt{2\pi\sigma_G^2\, t}} exp\left\{ -\frac{\left(x - \langle G \rangle t\right)^2}{2\sigma_G^2\, t} \right\}.$$

Step 2 – The first passage time of $x(t)$ to a boundary is shown to have an inverse Gaussian distribution with the density:

$$R\left(\langle x_{pt} \rangle, s\right) = \frac{\langle x_{pt} \rangle}{\sqrt{2\pi\sigma_G^2 s^3}} exp\left\{ -\frac{\left(\langle x_{pt} \rangle - \langle G \rangle s\right)^2}{2\sigma_G^2 s} \right\},\ s \geq 0.$$

Step 3 – The probability $\beta_{pt}(z)$ is evaluated as $Prob\left\{OT_{pt} \leq z\right\} = \int_z^{\infty} R\left(\langle x_{pt} \rangle, s\right) ds$, and the closed form approximation given in Proposition 1 is derived.

■

As the random operation time corresponds to a first passage time, it has been characterized by an inverse Gaussian distribution. The later is asymmetric and it is mostly skewed to the right, which corresponds to the qualitative representation existing in the literature: see for example (Vandaele and De Boeck, 2003). This means that if something happens in the system, the impact is dominantly negative: longer lead times will be observed more often than shorter ones (the average is larger than the mode). Since we are interested in the higher robust service levels, automatically this skewness is under the consideration of the management, and it will exploited to improve the probability of β_{pt}.

Proposition 2. Let denote by ***S*** the set of feasible solutions for the capacitated lot-sizing model defined by Equations (15)-(21). The following hold:

1. For every solution $s \in \boldsymbol{S}$, the probability β_{pt} of meeting a target service level SL_{pt}^* is equal to $\frac{1}{2}$.
2. If $\beta_{pt}* > \frac{1}{2}$, the set ***S*** is empty.

Proof. For every feasible solution ($s \in \boldsymbol{S}$), the operation times are the expected values of random variables OT_{pt} $(p \in P,\ t = 1,\ 2,\ ...,\ T)$: $\langle OT_{pt} \rangle = \dfrac{\langle x_{pt} \rangle}{\langle G \rangle}$. This means that the system produces during a time $\langle OT_{pt} \rangle$ to complete the lot size $\langle x_{pt} \rangle$. To evaluate the corresponding probability β_{pt}, the variable z is replaced by $\langle OT_{pt} \rangle$ in Equation (26) of Proposition 1. This gives β_{pt} = $\dfrac{1}{2}$, since $Erfc(0) = 1$. Consequently, if $\beta_{pt}* > \dfrac{1}{2}$, there is no feasible solution and $\boldsymbol{S}$ is empty.

■

When the variance of the production rate is neglected, the probabilities β_{pt} are equal to 100%. But, in our robust optimization context, this variance is of a central interest. On the one hand, it is indeed very unsatisfactory to use a production plan with a probability of meting a pre-specified service level that is equal to 50% (as shown by Proposition 2). On the other hand, increasing the operation time with only a few percents is sufficient to guarantee success with a high probability. This is due to the complementary error function Erfc(x) present in Equation (26) of proposition 1: see Figure 1. To illustrate this, let us consider a system with $CV_{OT_{pt}}{}^2 = \dfrac{\sigma_G{}^2}{\langle x_{pt} \rangle \ \langle G \rangle}$ = 0.05 and impose $\beta_{pt} = 0.95$. To obtain this value in Equation (26), we need to have $\dfrac{\langle G \rangle z}{\langle x_{pt} \rangle} \approx$ 0.7. Therefore, it is needed to produce during the time $z \approx \dfrac{\langle x_{pt} \rangle}{0.7 \langle G \rangle} \approx 1.43 \ \dfrac{\langle x_{pt} \rangle}{\langle G \rangle}$ to guarantee a quality criterion of $\beta_{pt} = 0.95$. Note that to have $\beta_{pt} = 0.9$, we obtain by following the same procedure, $z \approx 1.04 \ \dfrac{\langle x_{pt} \rangle}{\langle G \rangle}$. This means that increasing the operation time with only 4% is sufficient to guarantee meeting service level with a probability of $\beta_{pt} = 0.9$.

Examples of the probability β_{pt} as a function of $\dfrac{z}{\langle OT_{pt} \rangle}$ are shown in Figure 2. It is clear from this figure that, for diffusive regimes, the ratio $\dfrac{z}{\langle OT_{pt} \rangle}$ needs to be slightly larger than unity to guarantee that the promised service level will be satisfied with a high probability. Furthermore, for smaller lot sizes $\langle x_{pt} \rangle$ the influence of the fluctuations is more important and the probability β_{pt} is lower.

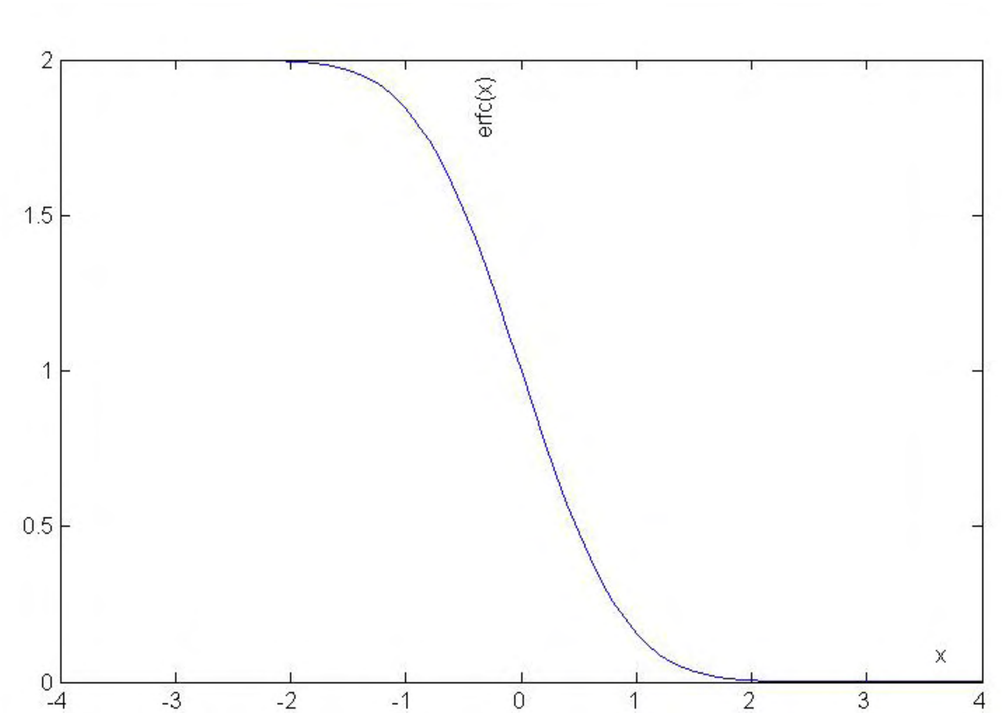

Figure 1. The complementary error function.

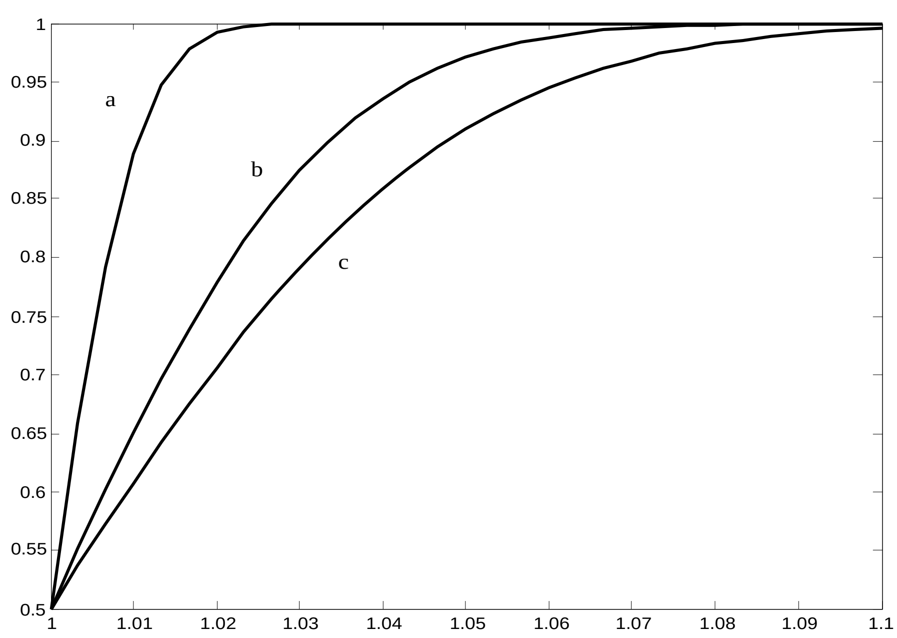

Figure 2. The probability β_{pt} as a function of the ratio $\dfrac{z}{\langle OT_{pt} \rangle}$: $\langle x_{pt} \rangle_a > \langle x_{pt} \rangle_b > \langle x_{pt} \rangle_c$.

3.2. *Improvement of the probability to meet a service level*

Two cases are distinguished: (1) when $\beta_{pt}*$ is less than 100%, an incremental procedure is proposed to improve gradually the probability β_{pt}; (2) when it is required to have a fully reliable service level, *i.e.*, $\beta_{pt} \square 1$, another procedure is used. Both procedures result from Proposition 1 as described below.

3.2.1. *Incremental improvement*

It is usually required to have $\beta_{pt}* > \frac{1}{2}$. However, if we solve the equivalent mean-value deterministic model of the problem formulated by (15)-(21), the set of feasible solutions will be empty. Motivated by the results of the previous subsections, we propose here an improvement procedure of the probability to meet a service level that acts on the optimal solution of the mean-value deterministic model solved without the chance constraints (20). This procedure allows us to ensure these constraints at the basis of Proposition 1.

From Equation (26) and Figure 1, we remark that it is possible to increase the probability β_{pt}, for a system characterised by its given $\langle G \rangle$ and σ_G, by increasing the effective operation time *z*. By doing so, the ratio $\frac{z}{\langle OT_{pt} \rangle}$ is increased and β_{pt} can be significantly improved, as previously shown by Figure 2. In our production planning problem, this will happen when $\langle G \rangle > \sum_{p \in P} \frac{\langle x_{pt} \rangle}{\langle OT_{pt} \rangle}$. In this case, the service levels are of 100% and the objective of increasing slightly a ratio $\frac{z}{\langle OT_{pt} \rangle}$ is to ensure a full service level with a higher probability. More precisely, the operation time $\langle OT_{pt} \rangle$ is increased by $k\ \Delta t$. The parameter *k* varies from 0 to a maximum value fixed by the difference between period length T_t and the operation time $\langle OT_{pt} \rangle$. This means simply that we produce a bit more $\left(x_{pt} = \langle x_{pt} \rangle + \langle G \rangle\ \Delta t \right)$ to increase the probability β_{pt}. The value of Δt corresponds to the time needed to produce one part $\left(\Delta t = \frac{1}{\langle G \rangle} \right)$. The final cost is evaluated for the updated solution after running the same procedure for all products and periods.

If $\langle G \rangle \leq \sum_{p \in P} \frac{\langle x_{pt} \rangle}{\langle OT_{pt} \rangle}$, the system functions during all the period, and another way to increase β_{pt} consists in decreasing the promised service level. That is, the new average operation time needed to produce the reduced promised lot size is smaller than $\langle OT_{pt} \rangle$. Therefore, producing during $\langle OT_{pt} \rangle$ (to accumulate such a reduced quantity) may result in a considerable improvement of β_{pt}. That is, for each product and each period, if the mean-value deterministic model has a service level SL_{pt} that is higher than the proposed target $SL_{pt}*$, the lot size $\langle x_{pt} \rangle$ is decreased by the quantity $\Delta B_{pt} = B_{pt}* - \langle B_{pt} \rangle$, without violating the constraints $SL_{pt} \geq SL_{pt}*$, and the new improved β_{pt} is re-evaluated.

It is possible to have some pairs $\{p,\ t\}$ for which the constraints $\beta_{pt} \geq \beta_{pt}*$ cannot be achieved. This happens when no remaining operation time is available and the service level is equal to the target value. In this case, to get feasible and robust solutions, it is necessary to improve the system characteristics to have a smaller variation coefficient of the production rate, to relax the service level constraints, or to increase lead times.

It is important to note that Proposition 1 has shown that only slight modifications are sufficient to reach robust solutions. As a result, substantial improvements in service level robustness are often possible with minimal increases in expected cost.

3.2.2. *Full improvement procedure*

The following propositions give necessary conditions to reach fully reliable service levels.

Proposition 3. For every period t and every product p, if the effective operation time z used to produce the lot size $\langle x_{pt} \rangle$ is increased by $4\frac{\sigma_G}{\langle G \rangle}\left(4\frac{\sigma_G}{\langle G \rangle}+\sqrt{2\langle OT_{pt} \rangle + 4\frac{\sigma_G{}^2}{\langle G \rangle^2}}\right)$, then β_{pt} □ 1.

Proof. Since $Erfc(2)$ □ 0 (see Figure 1), by using equation (26) in Proposition 1, we have:

if $\frac{\langle G \rangle}{\sigma_G}\sqrt{\frac{z}{2}}\left(1-\frac{\langle x_{pt} \rangle}{\langle G \rangle z}\right)=2$, then β_{pt} □ 1.

The only unknown in the above equation is the variable z, which is obtained by solving the following second order equation obtained after some simple manipulations:

$$z^2 + z\left(2\langle OT_{pt} \rangle + 8\frac{\sigma_G{}^2}{\langle G \rangle^2}\right) + \langle OT_{pt} \rangle^2 = 0 .$$

Knowing that z is positive, we obtain: $z = \langle OT_{pt} \rangle + 4\frac{\sigma_G}{\langle G \rangle}\left(4\frac{\sigma_G}{\langle G \rangle}+\sqrt{2\langle OT_{pt} \rangle + 4\frac{\sigma_G{}^2}{\langle G \rangle^2}}\right)$.

■

Proposition 4. For every period t and every product p, if the effective operation time is equal to $\langle OT_{pt} \rangle$ and the promised service level is decreased by a value corresponding to a reduced lot size of $\langle x_{pt} \rangle - 2\sigma_G\sqrt{2\langle OT_{pt} \rangle}$, then β_{pt} □ 1.

Proof. If the effective operation time is equal to $\langle OT_{pt} \rangle$ and the promised service level is decreased by a value τ such as $\frac{\langle G \rangle}{\sigma}\sqrt{\frac{\langle OT_{pt} \rangle}{2}}\left(1-\frac{\langle OT_{pt} \rangle - \tau}{\langle OT_{pt} \rangle}\right)=2$, then β_{pt} □ 1. The only unknown τ is obtained by solving this equation as $\tau = 2\frac{\sigma}{\langle G \rangle}\sqrt{2\langle OT_{pt} \rangle}$.

■

4. Numerical example

Let us consider a system having an average global production rate corresponding to 950 items/period, with $\sigma_G = 20$ items^2/period. The planning horizon H is 5 periods. The system has to produce two kinds of products in lots so that the demands are satisfied. For each product, the periodic demands are presented in Table 1. Table 2 gives the holding, backorder, set-up and production costs for each product. These costs are the same for all periods.

Table 1: Demands of products

Period	Demand of product 1 d_{1t} (items)	Demand of product 2 d_{2t} (items)
1	500	500
2	480	490
3	480	500
4	470	470
5	480	480

Table 2: Cost data of products

Product	Holding cost ($)	Backorder cost ($)	Set-up cost ($)	Production cost ($)
1	40	120	500	70
2	40	120	500	70

We consider that it is required to have $SL_{pt} \geq 0.89$ and $\beta_{pt} \geq 0.99$. By solving the mean-value deterministic model without the constraints $\beta_{pt} \geq 0.99$, we obtain the optimal production plan and its corresponding service levels as given in Table 3. The total cost for this plan is 386700 $.

Table 3: Optimal production plan

Product 1

Period	Production	Setup	Inventory	Backorder	Service level
1	450	1	0	50	0.9
2	478	1	0	52	0.89
3	487	1	0	45	0.9
4	464	1	0	51	0.89
5	479	1	0	52	0.89

Product 2

Period	Production	Setup	Inventory	Backorder	Service level
1	500	1	0	0	1
2	472	1	0	18	0.96
3	463	1	0	55	0.89
4	486	1	0	39	0.92
5	471	1	0	48	0.9

However, from Proposition 2, each probability β_{pt} is equal to 50%. In order to satisfy the constraints $\beta_{pt} \geq 0.99$, an improvement procedure is then applied. Since $\langle G \rangle \leq \sum_{p \in P} \frac{\langle x_{pt} \rangle}{\langle OT_{pt} \rangle}$, the system functions during all the period and the effective operation time cannot be increased. As it is required to have $\beta_{pt} \approx 1$, Proposition 4 is used. Let consider that the manager wants to keep the service levels in Table 3. That is, for every period *t* and every product *p*, the effective operation time is kept equal to $\langle OT_{pt} \rangle$ and the promised service level is decreased by a value corresponding to a reduced lot size of $\langle x_{pt} \rangle - 2\sigma_G \sqrt{2\langle OT_{pt} \rangle}$ denoted by $\langle \tilde{x}_{pt} \rangle$ in Table 4, where the promised (reduced) service level is also denoted by $\tilde{SL}_{pt}$. Table 4 confirms that slight modifications are sufficient to reach solutions with robust service levels.

Table 4: Robust service level solution

Product 1

Period	Production	$\langle \tilde{x}_{pt} \rangle$	$\tilde{SL}_{pt}$
1	450	411	0.82
2	478	437	0.91
3	487	446	0.93
4	464	426	0.91
5	479	438	0.91

Product 2

Period	Production	$\langle \tilde{x}_{pt} \rangle$	$\tilde{SL}_{pt}$
1	500	458	0.92
2	472	432	0.88
3	463	423	0.85
4	486	445	0.95
5	471	431	0.9

5. Conclusion

In this paper, we studied a multi-period, multi-product production planning problem where machines are subjected to random failures and repairs. As a result, the production rate and the customer service level are random variables. In the proposed model, constraints were introduced to ensure that a pre-specified customer service level is met with high probability. A two-step optimization approach was proposed to solve this model. It incorporates, with a mean value deterministic model, an improvement method of the service level robustness. The obtained results are in accordance with the existing literature. It is shown that substantial improvements in service level robustness are often possible with minimal increases in expected cost. The importance of our contribution lies in the fact that, unlike classical stochastic optimization and robust optimization approaches, the proposed method is *not* a scenario-based one. Instead of using a number of generated scenarios, the proposed optimization approach is based on the first passage time theory of a Wiener process to a boundary to take into account machine breakdowns. The resulting approach has then the advantage of being computationally tractable. The reason of making this possible is, of course, that the underlying process governing the behaviour of the machines has been characterized, in a way that has allowed the use of the first passage time theory in the context of drifted Brownian motion and inverse Gaussian distribution.

The application of the method to a real-life case study remains a perspective to investigate in a future work. For example, the results of the paper could be applied for sawmill production planning as in (Kazemi Zanjani et al., 2009, 2010). This necessitates data to characterize the random process of the sawing units. Future work will consider also the generalization of the proposed approach to deal with others sources of randomness such as yield, demand and prices. Furthermore, we are currently working on the integration of preventive maintenance planning into the production planning model developed in this paper.

Appendix – *Detailed proof of Proposition 1*

Step 1 – For asymptotically large times, invoking the central limit theorem, the cumulative production $\left\{x(t),\ t\geq 0 \middle| x(0)=x_0\right\}$ can be considered as a temporally homogeneous diffusion process following a stochastic differential equation of the form:

$$\frac{d}{dt}x_{pt}(t)=\langle G\rangle+\sigma_G\ \xi(t)\ ;\quad x(t=0)=0 \text{ and } G(0)=g\ ,$$

where $\xi(t)$ is a stochastic process in the form of a white Gaussian Noise (WGN) of zero mean and unit variance. The fluctuations of $G(t)$ can be characterized by a normal (Gaussian) law $N\left(\langle G\rangle,\ \sigma_G{}^2\right)$. The above equation is a *Langevin* equation, a type of stochastic differential equations well known in the literature of mathematic physics (Gardiner, 1983).

The time evolution of $x(t)$ constitutes a (Markovian) diffusion process on $\mathbb{R}$, and the transition probability density $Prob\left\{x\leq x(t)\leq x+dx\right\}dx=P\left(x,t\,|\,x_0,0\right)dx$ obeys to an associated forward *Fokker-Planck* (FP) equation (Gardiner, 1983), also known as *Chapman-Kolmogorov*:

$$\frac{\partial}{\partial t}P\left(x,t\,|\,x_0,0\right)=-\langle G\rangle\ \frac{\partial}{\partial x}P\left(x,t\,|\,x_0,0\right)+\frac{\sigma_G{}^2}{2}\frac{\partial^2}{\partial x^2}P\left(x,t\,|\,x_0,0\right).$$

Due to the linearity of this equation, one immediately has:

$$P\left(x,t\,|\,x_0,0\right)=\frac{1}{\sqrt{2\pi\sigma_G{}^2\ t}}\,exp\left\{-\frac{\left(x-\langle G\rangle t\right)^2}{2\sigma_G{}^2\ t}\right\}.$$

Having $P\left(x,t\,|\,x_0,0\right)=P\left(x,t\,|\,0,0\right)$, we adopt the notation $P\left(x,t\,|\,x_0,0\right)=P\left(x-x_0,t\right)=P\left(x,t\right)$.

Step 2 – The calculation of β_{pt} relies on the determination of the first passage time of the stochastic process $x_{pt}(t) = x(t)$ to the boundary $\langle x_{pt} \rangle$, knowing the initial position (at t = 0) is x_0. The random time OT_{pt} needed to complete a batch of size $\langle x_{pt} \rangle$ is:

$$OT_{pt} = \inf_{t>0} \left\{ t : x(t) \geq \langle x_{pt} \rangle \right\} \text{, with } x(0) = x_0 < \langle x_{pt} \rangle .$$

The associated probability density is written as:

$$R\left(\langle x_{pt} \rangle, s \mid x_0, 0 \right) ds = Prob\left\{ s \leq OT_{pt} \leq s + ds \right\} .$$

We have $R\left(\langle x_{pt} \rangle, s \mid x_0, 0 \right) = R\left(\langle x_{pt} \rangle - x_0, s \mid 0, 0 \right)$, or simply $R\left(\langle x_{pt} \rangle, s \mid 0, 0 \right) = R\left(\langle x_{pt} \rangle, s \right)$.
The probability density $R\left(\langle x_{pt} \rangle, s \right)$ can be given by solving the following equation (Siegert, 1951; Darling and Siegert, 1953):

$$P(x,t) = \int_0^t R\left(\langle x_{pt} \rangle, s \right) P\left(x - \langle x_{pt} \rangle, t - s \right) ds .$$

The above equation expresses the fact that the probability to transit form position 0 to position x at time t can be expressed by the probability to reach $\langle x_{pt} \rangle$ at time s first (with $s \leq t$ and $x \geq \langle x_{pt} \rangle$) and then from $\langle x_{pt} \rangle$ to reach x in the remaining allotted time (t-s). This equation being a convolution, it can be solved by Laplace transformation which leads to an inverse Gaussian distribution with the density (Seshadri, 1993):

$$R\left(\langle x_{pt} \rangle, s \right) = \frac{\langle x_{pt} \rangle}{\sqrt{2\pi \sigma_G{}^2 s^3}} exp\left\{ -\frac{\left(\langle x_{pt} \rangle - \langle G \rangle s \right)^2}{2\sigma_G{}^2 s} \right\}, \ s \geq 0.$$

Step 3 – The probability $\beta_{pt}(z)$ is evaluated as $Prob\left\{ OT_{pt} \leq z \right\} = \int_z^\infty R\left(\langle x_{pt} \rangle, s \right) ds$. Before approximating this integral, let us calculate the average and the variance of the operation time OT_{pt}, *i.e.*, the time to complete a lot of size $\langle x_{pt} \rangle$. They are given by:

$$\langle OT_{pt} \rangle = \int_0^\infty t \ \left(R\langle x_{pt} \rangle, t \right) \ dt \ , \ \sigma_{OT_{pt}}{}^2 = \int_0^\infty t^2 \ R(\langle x_{pt} \rangle, t) \ dt - \langle OT_{pt} \rangle^2 .$$

We can straightforwardly obtain the following equations:

$$\langle OT_{pt} \rangle = \frac{\langle x_{pt} \rangle}{\langle G \rangle} , \ \sigma_{OT_{pt}}{}^2 = \frac{\sigma_G{}^2}{\langle G \rangle^3} \langle x_{pt} \rangle .$$

The squared coefficient of variation for the time OT_{pt} is:

$$CV_{OT_{pt}}{}^2 = \frac{\sigma_{OT_{pt}}{}^2}{\left\langle OT_{pt} \right\rangle^2} = \frac{\sigma_G{}^2}{\left\langle x_{pt} \right\rangle \left\langle G \right\rangle}.$$

Finally, we can write:

$$\beta_{pt}\left(z\right) = \int_z^{\infty} \frac{\left\langle x_{pt} \right\rangle}{\sqrt{2\pi\sigma_G{}^2 s^3}} exp\left\{ -\frac{\left(\left\langle x_{pt} \right\rangle - \left\langle G \right\rangle s \right)^2}{2\sigma_G{}^2 s} \right\} ds .$$

Let us introduce the following notations: $a\left(z\right) = \Gamma \frac{z}{\left\langle OT_{pt} \right\rangle} \left(1 - \frac{\left\langle OT_{pt} \right\rangle}{z} \right)^2, z \geq OT_{pt}$ with $\Gamma = \frac{1}{CV_{OT_{pt}}}$.

Using Shuster's derivation (Shuster, 1968), the above equation can be rewritten, for $z \geq \left\langle OT_{pt} \right\rangle$, in the form:

$$\beta_{pt}(z) = 1 - \frac{1}{2} Erfc\left(\sqrt{\frac{a(z)}{2}} \right) + \frac{1}{2} e^{2\Gamma} \cdot Erfc\left(\sqrt{\frac{a(z)}{2} + 2\Gamma} \right) \text{ with } Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)\, dt .$$

Considering the case where the magnitude of the fluctuations is such that the coefficient of variation $CV_{OT_{pt}}$ is relatively small (i.e., $2\Gamma >> 1$), and knowing from (Abramowitz and Stegun, 1964) that for $x>0$, $Erfc(x)$ is bounded such as $\left(\frac{2}{\sqrt{\pi}} \right) \frac{e^{-x^2}}{x + \sqrt{x^2 + 2}} < Erfc(x) \leq \left(\frac{2}{\sqrt{\pi}} \right) \frac{e^{-x^2}}{x + \sqrt{x^2 + \frac{4}{\pi}}}$, the above equation of $\beta_{pt}(z)$ can be approximated by neglecting the last term, in the form of:

$$\beta_{pt}\left(z\right) \approx 1 - \frac{1}{2} Erfc\left(\frac{1}{CV_{OT_{pt}}} \sqrt{\frac{z}{2\left\langle OT_{pt} \right\rangle}} \left(1 - \frac{\left\langle OT_{pt} \right\rangle}{z} \right) \right),$$

which is equivalent to equation (26) in Proposition 1.

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