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Interuniversity Research Centre on Enterprise Networks, Logistics and Transportation

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September 2009

CIRRELT-2009-39

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Abstract. Multi-echelon distribution systems are quite common in supply-chain and logistic management. They are used by public administrations in their transportation and traffic planning strategies as well as by companies to model their distribution systems. In the literature, most studies address issues related to the movement of flows throughout the system from the origins to their final destinations. In this paper we consider the Two-Echelon Capacitated Vehicle Routing Problem (2E-CVRP), the two-echelon variant of the well known Capacitated Vehicle Routing Problem, where the delivery from one depot to the customers is managed by routing and consolidating freight through intermediate depots, called satellites. The main goal of this paper is the definition of new classes of valid inequalities for strengthening the linear formulation of the 2E-CVRP. More in detail, valid inequalities based on the TSP and CVRP, the network flow formulation, and the connectivity of the transportation system graph are presented. These valid inequalities are tested through a branch-and-cut algorithm and extensive computational results on instances with up to 50 customers and 5 satellites are reported.

Keywords. 2E-CVRP, valid inequalitie, branch-and-cut.

Acknowledgements. This project has been partially supported by the Ministero dell'Università e della Ricerca (MIUR) (Italian Ministery of Education, University and Research) under the “Progetto di Ricerca di Interesse Nazionale” (PRIN) 2007 Optimization of Distribution Logistics. Funding has also been provided by the Natural Sciences and Engineering Research Council of Canada (NSERC), through its Industrial Research Chair and Discovery Grants programs, by the partners of the Chair, CN, Rona, Alimentation Couche-Tard and the Ministry of Transportation of Québec, and by the Fonds de recherche sur la nature et les technologies (FQRNT) through its Team Research Grants program.

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Dépôt légal – Bibliothèque nationale du Québec,
Bibliothèque nationale du Canada, 2009

In Multi-Echelon Vehicle Routing Problems the delivery from one or more depots to the customers is managed by routing and consolidating the freight through intermediate depots, called satellites. This family of problems differs from Multi-Echelon systems present in the literature, where the attention is focused on the flow assignment among the levels only, while in our case, we also consider the fleet management and the overall distribution system routing. The impact of the routing in Multi-Echelon distribution Systems is strictly connected to the City Logistics aspect. In fact, in this way we can consider environmental aspects, as well as plan the effect of keeping big trucks far from the city, while using small and environmental friendly vehicles for the deliveries in the historical city centers [25]. In this work we deal with the Two-Echelon Capacitated Vehicle Routing Problem (2E-CVRP), the basic variant of Multi-Echelon Vehicle Routing with one depot and a fixed number of satellites. First level routing deals with depot-to-satellites delivery, while the second level one with satellites-to-customers delivery. The fleet is fixed and homogeneous among each level. Vehicles and satellites are capacitated, while neither synchronization between the vehicles in each satellite nor time-windows for the customers are considered.

The literature on 2E-CVRP is limited, due to the recent introduction of the problem itself. A model for the 2E-CVRP and two families of valid inequalities, by which instances up to 32 customers can be solved to the optimum, have been presented in [25]. In the same work, the authors derived two math-heuristics able to solve instances up to 50 customers. Concerning larger size instances, a fast cluster-based heuristic method able to deal with instances up to 150 customers has been proposed by Crainic et al. [11]. For an application of Two-Echelon VRP on freight distribution system we refer the reader to [10] in which advanced freight distribution systems are provided. The authors investigate the City Logistic Planning Problem, propose a general model and formulations for the main system components and identify promising solution trends.

The main goal of this work consists in detecting and defining new classes of valid inequalities for strengthening existent linear formulation during 2E-CVRP optimization. More in detail, valid inequalities derived from the literature of the Traveling Salesman Problem (TSP) and the Capacitated Vehicle Routing Problem (CVRP), from the MIP formulation as well as the connectivity of the transportation system graph are presented.

The paper is organized as follows. In Section 1 the MIP model is recalled, while new valid inequalities are presented in Section 2. Section 3 is devoted to present the effectiveness of the new cuts in solving to optimality the problem, while Section 4 summarizes the results and presents some future trends in the research on 2E-CVRP.

1 Mathematical model

Let us denote the depot by v_0, the set of intermediate depots, called satellites, by V_s and the set of customers by V_c. Let n_s be the number of satellites and n_c the number of customers. The depot is the starting point of the freight and the satellites are capacitated. The customers are the destinations of the freight and each customer i has associated a demand d_i, i.e. the quantity of freight that has to be delivered to that customer. The demand of each customer cannot be split among different vehicles at the 2nd level. For the first level, we consider that each satellite can be served by more than one 1st-level vehicle, so the aggregated freight assigned to each satellite can be split into two or more vehicles. Each 1st level vehicle can deliver the freight of one or more customers, as well as serve more than one satellite in the same route.

The distribution of the freight cannot be managed by direct shipping from the depot to the customers. Instead the freight must be consolidated from the depot to a satellite and then delivered from the satellite to the desired customer. This implicitly defines a two-echelon transportation system: the 1st level interconnecting the depot to the satellites and the 2nd one the satellites to the customers (see Figure 1). Define the arc (i, j) as the direct

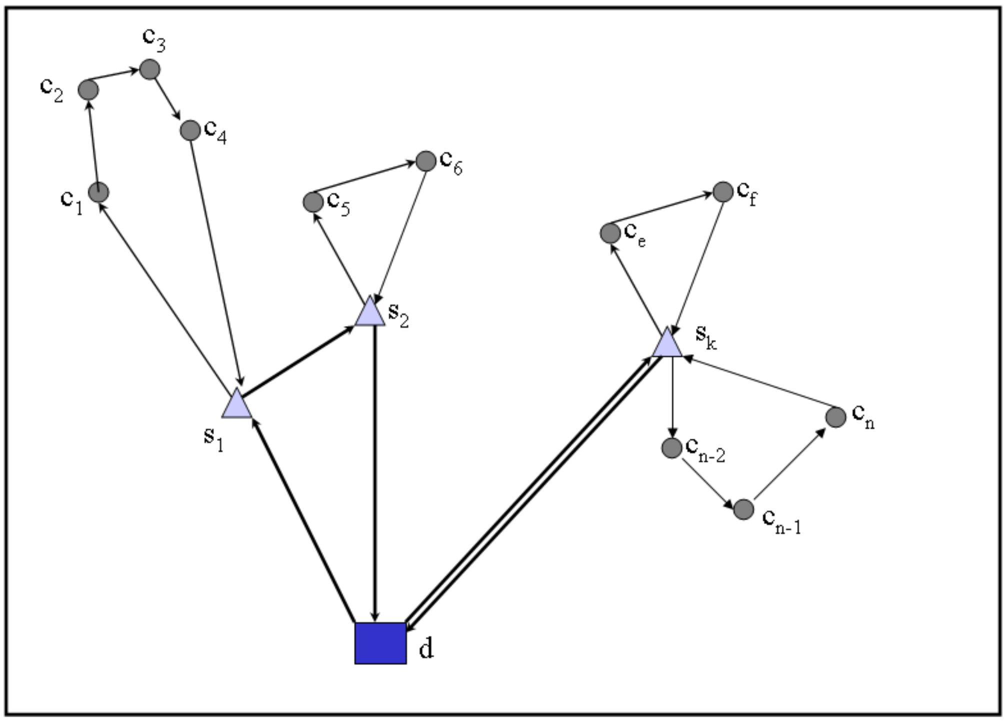

Figure 1: Example of 2E-CVRP transportation network

route connecting node i to node j. If both nodes are satellites or one is the depot and the other is a satellite, we define the arc as belonging to the 1st-level network, while if both nodes are customers or one is a satellite and the other is a customer, the arc belongs to the 2nd-level network.

We define as *1st-level route* a route made by a 1st-level vehicle which starts from the depot, serves one or more satellites and ends at the depot. A *2nd-level route* is a route made by a 2nd-level vehicle which starts from a satellite, serves one or more customers and ends at the same satellite.

The freight must be delivered from the depot v_0 to the customers set $V_c = \{v_{c_1}, v_{c_2}, ..., v_{c_{n_c}}\}$. Let d_i be the demand of the customer c_i. The number of 1st-level vehicles available at the

$V_0 = \{v_0\}$	Depot
V_s	Set of satellites
V_c	Set of customers
n_s	Number of satellites
n_c	Number of customers
m_1	Number of the 1st-level vehicles
m_2	Number of the 2nd-level vehicles
m_{s_k}	Maximum number of 2nd-level routes starting from satellite k
K^1	Capacity of the vehicles for the 1st level
K^2	Capacity of the vehicles for the 2nd level
d_i	Demand required by customer i
c_{ij}	Cost of the arc (i, j)
F_k	Cost for loading/unloading operations of a unit of freight in satellite k
Q^1_{ij}	Flow passing through the 1st-level arc (i, j)
Q^2_{ijk}	Flow passing through the 2st-level arc (i, j) and coming from satellite k
x_{ij}	Number of 1st-level vehicles using the 1st-level arc (i, j)
y^k_{ij}	Boolean variable equal to 1 if the 2st-level arc (i, j) is used by the 2nd-level routing starting from satellite k
z_{kj}	Variable set to 1 if the customer c_i is served by the satellite k

Table 1: Definitions and notations

depot is m_1. These vehicles have the same given capacity K^1. The total number of 2nd-level vehicles available for the second level is equal to m_2. The total number of active vehicles can not exceed m_2 and each satellite k has a maximum capacity m_{s_k}. The 2nd-level vehicles have the same given capacity K^2. No additional limitation on the route size, neither in length nor in number of visited customers is introduced.

In our model we will not consider the fixed costs of the vehicles, since we suppose they are available in fixed number. We consider the travel costs c_{ij}, which are of two types:

- costs of the arcs traveled by 1st-level vehicles, i.e. arcs connecting the depot to the satellites and the satellites between them;
- costs of the arcs traveled by 2nd-level vehicles, i.e. arcs connecting the satellites to the customers and the customers between them.

Another cost that can be used is the cost of loading and unloading operations at the satellites. Supposing that the number of workers in each satellite k is fixed, we consider only the cost incurred by the management of the freight and we define F_k as the unit cost of freight handling at the satellite k.

The formulation we present derives from the multi-commodity network design and uses the flow of the freight on each arc as main decision variables.

We define five sets of variables, that can be divided in three groups:

- The first group represents the arc usage variables. We define two sets of such variables, one for each level. The variable x_{ij} is an integer variable of the 1st-level routing and is equal to the number of 1st-level vehicles using arc (i, j). The variable y_{ij}^k is a binary variable representing the 2nd-level routing. It is equal to 1 if a 2nd-level vehicle makes a route starting from satellite k and goes directly from node i to node j, 0 otherwise.

- The second group of variables represents the assignment of each customer to one satellite and are used to link the two transportation levels. More precisely, we define z_{kj} as a binary variable that is equal to 1 if the freight to be delivered to customer j is consolidated in satellite k and 0 otherwise.
- The third group of variables, split into two subsets, one for each level, represents the freight flow passing through each arc. We define the freight flow as a variable Q_{ij}^1 for the 1st-level and Q_{ijk}^2 for the 2nd level, where k represents the satellite where the freight is passing through. Both variables are continuous.

In order to lighten the model formulation, we define the auxiliary quantity

$$D_k = \sum_{j \in V_c} d_j z_{kj}, \forall k \in V_s, \tag{1}$$

which is non-negative and represents the freight passing through each satellite k.

The model to minimize the total cost of the system may be formulated as follows (please refer to Table 1 for definitions and notation):

$$\min \sum_{i,j \in V_0 \cup V_s, i \neq j} c_{ij} x_{ij} + \sum_{k \in V_s} \sum_{i,j \in V_s \cup V_c, i \neq j} c_{ij} y_{ij}^k + \sum_{k \in V_s} F_k D_k \tag{2}$$

$$\sum_{i \in V_s} x_{0i} \leq m_1 \tag{3}$$

$$\sum_{j \in V_s \cup V_0, j \neq k} x_{jk} = \sum_{i \in V_s \cup V_0, i \neq k} x_{ki} \quad \forall k \in V_s \cup V_0 \tag{4}$$

$$\sum_{k \in V_s} \sum_{j \in V_c} y_{kj}^k \leq m_2 \tag{5}$$

$$\sum_{j \in V_c} y_{kj}^k \leq m_{s_k} \quad \forall k \in V_s \tag{6}$$

$$\sum_{j \in V_c} y_{kj}^k = \sum_{j \in V_c} y_{jk}^k \quad \forall k \in V_s \tag{7}$$

$$\sum_{i \in V_s \cup v_0, i \neq j} Q_{ij}^1 - \sum_{i \in V_s \cup v_0, i \neq j} Q_{ji}^1 = \begin{cases} D_j & j \text{ is not the depot} \\ \sum_{i \in V_c} -d_i & \text{otherwise} \end{cases} \quad \forall j \in V_s \cup V_0 \tag{8}$$

$$Q_{ij}^1 \leq K^1 x_{ij} \quad \forall i, j \in V_s \cup V_0, i \neq j \tag{9}$$

$$\sum_{i \in V_c \cup k, i \neq j} Q_{ijk}^2 - \sum_{i \in V_c \cup k, i \neq j} Q_{jik}^2 = \begin{cases} z_{kj} d_j & j \text{ is not a satellite} \\ -D_j & \text{otherwise} \end{cases} \quad \forall j \in V_c \cup V_s, \forall k \in V_s \tag{10}$$

$$Q^2_{ijk} \leq K^2 y^k_{ij} \quad \forall i,j \in V_s \cup V_c, i \neq j, \forall k \in V_s \tag{11}$$

$$\sum_{i \in V_s} Q^1_{iv_0} = 0 \tag{12}$$

$$\sum_{j \in V_c} Q^2_{jkk} = 0 \quad \forall k \in V_s \tag{13}$$

$$y^k_{ij} \leq z_{kj} \quad \forall i \in V_s \cup V_c, \forall j \in V_c, \forall k \in V_s \tag{14}$$

$$y^k_{ji} \leq z_{kj} \quad \forall i \in V_s, \forall j \in V_c, \forall k \in V_s \tag{15}$$

$$\sum_{i \in V_s \cup V_c} y^k_{ij} = z_{kj} \quad \forall k \in V_s, \forall j \in V_c \tag{16}$$

$$\sum_{i \in V_s} y^k_{ji} = z_{kj} \quad \forall k \in V_s, \forall j \in V_c \tag{17}$$

$$\sum_{i \in V_s} z_{ij} = 1 \quad \forall j \in V_c \tag{18}$$

$$y^k_{kj} \leq \sum_{l \in V_s \cup V_0} x_{kl} \quad \forall k \in V_s, \forall j \in V_c \tag{19}$$

$$y^k_{ij} \in \{0,1\}, \quad \forall k \in V_s \cup V_0, \forall i,j \in V_c \tag{20}$$

$$z_{kj} \in \{0,1\}, \quad \forall k \in V_s \cup V_0, \forall j \in V_c \tag{21}$$

$$x_{kj} \in \mathbb{Z}^+, \quad \forall k,j \in V_s \cup V_0 \tag{22}$$

$$Q^1_{ij} \geq 0, \forall i,j, \in V_s \cup V_0,\ Q^2_{ijk} \geq 0,\ \forall i,j \in V_s \cup V_c, \forall k \in V_s. \tag{23}$$

The objective function minimizes the sum of the traveling and handling operations costs. Constraints (4) show, for $k = v_0$, that each 1st-level route begins and ends at the depot, while when k is a satellite, impose the balance of vehicles entering and leaving that satellite. The limit on the satellite capacity is satisfied by constraints (6). They limit the maximum number of 2nd-level routes starting from every satellite (notice that the constraints also limit at the same time the freight capacity of the satellites). Constraints (7) force each 2nd-level route to begin and end to one satellite and the balance of vehicles entering and leaving each customer. The number of the routes in each level must not exceed the number of vehicles for that level, as imposed by constraints (3) and (5).

Constraints (8) and (10) indicate that the flows balance on each node is equal to the demand of this node, except for the depot, where the outgoing flow is equal to the total demand of the customers, and for the satellites at the 2nd-level, where the flow is equal to the demand (unknown) assigned to the satellites. Moreover, constraints (8) and (10) forbid the presence of subtours not containing the depot or a satellite, respectively. The capacity constraints are formulated in (9) and (11), for the 1st-level and the 2nd-level, respectively. Constraints (12) and (13) do not allow residual flows in the routes, making the returning flow of each route to the depot (1st-level) and to each satellite (2nd-level) equal to 0.

Constraints (14) and (15) indicate that a customer j is served by a satellite k ($z_{kj} = 1$) only if it receives freight from that satellite ($y_{ij}^k = 1$). Constraint (18) assigns each customer to one and only one satellite, while constraints (16) and (17) indicate that there is only one 2nd-level route passing through each customer. At the same time, they impose the condition that a 2nd-level route departs from a satellite k to deliver freight to a customer if and only if the customer's freight is assigned to the satellite itself. Constraints (19) allow a 2nd-level route to start from a satellite k only if a 1st-level route has served it.

Finally, (20)-(23) specify the domains of the variables. In particular, notice that while the arc variables y_{ij}^k is defined as boolean, being each customer served by at most one route, the 1st-level arc variables x_{kj} must be integer. This is due to the fact that each satellite may be served by more than one vehicle and that the different vehicles may share the same arc.

2 Valid Inequalities for the 2E-CVRP

In the following, we present several families of valid inequalities for the 2E-CVRP. Some of them are an extension to the 2E-CVRP of existing cuts for TSP and VRP (see [6] for a survey), while others are directly related to the peculiar properties of 2E-CVRP.

In the following, we consider to have an optimal solution of the continuous relaxation of model (2)-(23) and we refer to the continuous model as $C - 2ECVRP$. Moreover, we will refer to variables with a non-zero value as *active variables*.

2.1 Generalization of existing valid inequalities

2.1.1 Valid Inequalities derived from TSP

In [25] the authors extend the Subtour Elimination Constraints for the TSP as

$$\sum_{i,j \in V'} y_{ij}^k \leq |V'| - 1, \quad \forall V' \subset V_c, \quad 2 \leq |V'| \leq |V_c| - 1, \quad k \in V_s. \tag{24}$$

According to the authors' results, no violation for subsets V' with more than 3 customers are present in the tested instances. A new set of potentially violated inequalities can be obtained grouping (24) on the set of satellites V_s, obtaining

$$\sum_{k \in V_s} \sum_{i,j \in V'} y_{ij}^k \leq |V'| - 1, \quad \forall V' \subset V_c, \quad 2 \leq |V'| \leq |V_c| - 1. \tag{25}$$

Valid inequalities (25) can be separated by finding subsets V' by recursive selection of y_{ij}^k between active variables only. If a cycle is found, then, constraint (25) is evaluated, by considering all arc variables related to nodes set V'. Notice that inequalities (25) can be violated even when inequalities (24) are not violated.

2.1.2 Valid Inequalities derived from CVRP

A large family of cutting planes derived from the CVRP can be considered, under certain hypothesis, as valid even for the 2E-CVRP.

Consider an instance of 2E-CVRP and focus on the 2nd-level network restricted to a given satellite k. Any solution of this network can be seen as a special case of CVRP, where the set of the nodes is not known in advance, due to the presence of the assignment variables z_{kj}. The presence of these assignment variables does not let to directly consider existing valid inequalities introduced in the CVRP literature.

Let us consider a graph $\tilde{G}$ where the set of the nodes is $\tilde{V} = V_c \cup \{l\}$, where l is a macro-node where we collapse the depot d_0 and all the satellites and the following auxiliary problem, which represents the 2E-CVRP restricted to the 2nd-level only:

$$\min \sum_{i,j \in \tilde{V}, i \neq j} \tilde{c}_{ij} w_{ij} \tag{26}$$

$$w(\delta(\{i\})) = 2 \quad \forall i \in V_c \tag{27}$$

$$w(\delta(\{l\})) \leq 2m_2 \tag{28}$$

$$\sum_{\substack{i \in \tilde{V} \\ i \neq j}} \mathrm{Q}_{ij} - \sum_{\substack{i \in \tilde{V} \\ i \neq j}} \mathrm{Q}_{ji} = \begin{cases} d_j & \text{if } j \neq l \\ \sum_{i \in \tilde{V}} -d_i & \text{otherwise} \end{cases} \quad \forall j \in \tilde{V} \tag{29}$$

$$\mathrm{Q}_{ij} \leq K^2 w_{ij} \quad \forall i, j \in \tilde{V}, i \neq j \tag{30}$$

$$w_{ij} \in \{0,1\}, \quad \mathrm{Q}_{ij} \geq 0, \quad \forall i, j \in \tilde{V}, i \neq j \tag{31}$$

where $w(\delta(V'))$ is the sum of all the w_{ij} variables such that associated edges (i,j) have a node incident in V' and the other one in $\tilde{V} \backslash V'$ and $\tilde{c}_{ij}$ are the costs associated to the usage of the arcs.

In the following, we will refer to the continuous relaxation of model (26)-(31) as $C{-}2LEV$.

Lemma 1. *Any feasible solution of $C - 2ECVRP$ is a feasible solution of $C - 2LEV$.*

Proof. Let we consider a feasible solution $\bar{x}$ of the $C - 2ECVRP$ polyhedron. Restricting only to 2nd level network variables, by constraints (16), (17) and (18),

$$\sum_{i \in \tilde{V}} w_{ij} = \sum_{k \in V_s} \sum_{i \in V_c} y_{ij}^k + \sum_{k \in V_s} y_{kj}^k = \sum_{k \in V_s} \sum_{i \in V_c} y_{ji}^k + \sum_{k \in V_s} y_{jk}^k = \sum_{i \in \tilde{V}} w_{ji} = 1 \quad \forall j \in V_c. \tag{32}$$

Since y_{ij}^k variables are boolean, $w_{ij} = \sum_{k \in V_s} y_{ij}^k \ \forall i, j \in V_c$, $w_{li} = \sum_{k \in V_s} y_{ki}^k \ \forall i \in V_c$ and $w_{jl} = \sum_{k \in V_s} y_{jk}^k \ \forall j \in V_c$ are also boolean. Similarly, constraints (18) guarantee the equivalence between (10) and (29) for Q_{ij} variables. Then, a feasible solution of $C - 2ECVRP$ is also a feasible solution of $C - 2LEV$. ◇

We remark that converse is not true. Any feasible solution in the polyhedron associated to $C-2LEV$ does not correspond to a unique solution, feasible or not, for $C-2ECVRP$: then there is no information about assignment variable z_{kj} values.

Let us consider the CVRP instance defined on $\tilde{G}$ where:

- node l is the depot;
- the demand d_j of each customer is the demand on the 2E-CVRP instance;
- the costs $\tilde{c}_{ij}$ are the original costs c_{ij} of the 2E-CVRP instance if both nodes are customers and 0 if one of them is l.

It is easy to see that the CVRP instance previously defined on $\tilde{G}$ is represented by model (26)-(31). Thus, any valid inequality for the CVRP that can be detected on graph $\tilde{G}$ can be converted in an similar valid inequality for $C-2ECVRP$ by

$$w_{ij} = \sum_{k \in V_s} y_{ij}^k. \tag{33}$$

Capacity Inequalities

A generic capacity inequality for 2E-CVRP can be written in the form

$$\sum_{k \in V_s} y^k\left(E(V')\right) \leq |V'| - r\left(V'\right) \quad \forall V' \subset V_c \quad 2 \leq |V'| \leq |V_c| - 1 \tag{34}$$

where, for commodity of use, we denote

$$y^k\left(E(V')\right) = \sum_{\substack{i,j \in V' \\ i \neq j}} y_{ij}^k \quad k \in V_s$$

or, by $C-2LEV$ notation, as

$$w\left(E(V')\right) \leq |V'| - r(V') \quad \forall V' \subset V_c \quad 2 \leq |V'| \leq |V_c| - 1, \tag{35}$$

where $r(V')$ is the optimal objective function of the Bin Packing instance where the size of the bins is equal to the second-level vehicle capacity and the items volume is the demand of the customers in V'.

Remarking the difficulty to separate capacity constraint class for the CVRP [9], we introduce the following Relaxed Capacity Inequalities (RCIs) by substituting $r(V')$ with the continuous lower bound for the Bin Packing

$$w\left(E(V')\right) \leq |V'| - \left\lceil \frac{d(V')}{K^2} \right\rceil \quad \forall V' \subset V_c \quad 2 \leq |V'| \leq |V_c| - 1, \tag{36}$$

where $d(V')$ denotes the sum of the demands of customers in V'. Capacity inequalities, even written in the weak form (36), still dominates edge cuts (24) and (25).

It is well known that, like the subtour elimination class, the number of candidates for RCIs violations is exponential in the size of the instance. In addition, theoretical results confirm that the difficulty in separation problem for these constraints is strongly *NP-hard* [27]. An exact separation algorithm for RCIs for VRP class, based on mixed integer models, can be found in [14].

In order to reduce the computational effort, many heuristic algorithms have been developed for separating of RCIs and special cases of (35). The largest part of them is based on the concept of *shrinking graph*, an idea formalized by Gomory and Hu [15], successfully used in the algorithm for the minimum capacity cut problem [23], and then applied to VRP context [4]. The idea consists in checking potential violations of target class of valid inequalities by considering a candidate set which is a connected component (or a part of it) of a support graph $G^* = (V_w^*, E_w^*)$. Thus, the shrinking operation consists in combining two vertices connected by a specific arc (i, j) with w_{ij} or w_{ji} with non-zero value in the relaxed solution solution of $C - 2LEV$, in order to form a single *super-vertex* (see Figure 2 for an example). The approach for checking 2E-CVRP RCIs is based on separation algorithms implemented for the classical CVRP problem by Lysgaard et al. in [20] (for further details, refer to [20, 5, 19]).

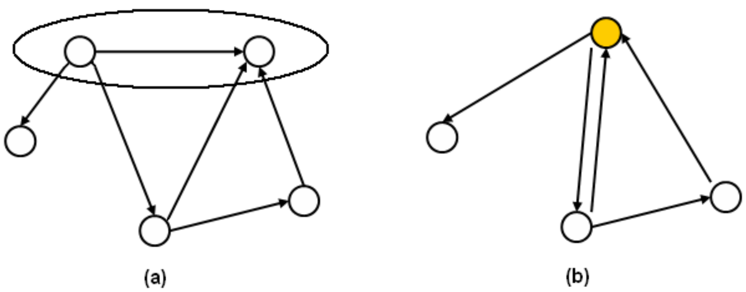

Figure 2: Shrinking procedure: candidate vertices **(a)**; shrunk graph **(b)**

Strengthened Comb Inequalities

Comb inequalities are a highly useful cutting plane class for TSP instances [21], involving connectivity violations on arc flows upon node subsets of customers in the graph representing the transportation network. Stated their effectiveness to improve TSP linear relaxed problem, several authors have attempted to introduce an equivalent form into VRP [18, 1].

A *comb* is defined as a list of $t+1$ node sets, one called *handle* $H \subset V_c$, and the remaining ones $T_1, ..., T_t \subset V_c \cup \{l\}$ named *teeth*, such that the following conditions hold [2]:

- $H \cap T_j \neq \emptyset$ and $T_j \backslash H \neq \emptyset$, for $j = 1, ..., t$;
- $T_i \cap T_j \subset H \vee T_i \cap T_j \cap H = \emptyset \; \forall i, j \in \{1, ..., t\}$, $i \neq j$.

Defined

$$S(H, T_1, ..., T_t) = \sum_{j=1}^{t} \left(\tilde{r}(T_j \cap H) + \tilde{r}(T_j \backslash H) + \tilde{r}(T_j) \right)$$

where $\tilde{r}(S)$ is equal to $r(S)$ if depot $0 \notin S$, $r(V \backslash S)$ otherwise, $\forall S \subset V$, if $S(H, T_1, ..., T_t)$ is odd, the related strengthened comb inequality is

$$w(\delta(H)) + \sum_{j=1}^{t} w(\delta(T_j)) \geq S(H, T_1, ..., T_t) + 1. \tag{37}$$

and it is valid for the 2E-CVRP (see Figure 3 for an example).

Inequalities (37) can be reduced to ordinary comb inequalities when $S(H, T_1, ..., T_t) = 3t$ and teeth intersections are not allowed.

Separating this class of valid inequalities is proven to be *NP-hard*, except for special cases [22]. A good heuristic uses the shrinking procedure mentioned before. Like in RCIs, $r(S)$ is replaced with its bound $k(S) = \lceil d(S)/K^2 \rceil$. The heuristic we use for the separation is the one presented in [20, 19] and the candidate sets selection is made considering criteria presented in [24].

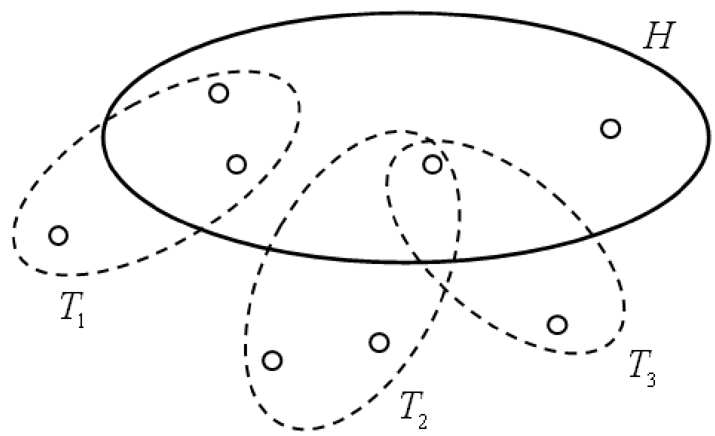

Figure 3: Example of a Comb Inequality with teeth $t = 3$

Multistar e Partial Multistar Inequalities

Given two disjoint vertex sets $V1$, $V2$, let $E(V1 : V2)$ denote the set of edges "crossing" from $V1$ to $V2$. More formally, $E(V1 : V2) = \delta(V1) \cap \delta(V2)$.

The *multistar* inequality class, originally defined by [3] for the CVRP problem with Unit Demand, can be written for 2E-CVRP as follows:

$$\alpha w(E(N)) + \beta w(E(N : V')) \leq \gamma \tag{38}$$

where $N \subset V_c$ is the *nucleus* set, $V' \subseteq V_c \backslash N$ is the *satellite set*, and α, β, γ are constants depending on $|N|$ and $|V'|$ (see Figure 4 for an example).

Another variant of inequality in the same class is the *partial multistar inequality*, also introduced in [3], and defined as follows

$$\alpha w(E(N)) + \beta w(E(C : V')) \leq \gamma \tag{39}$$

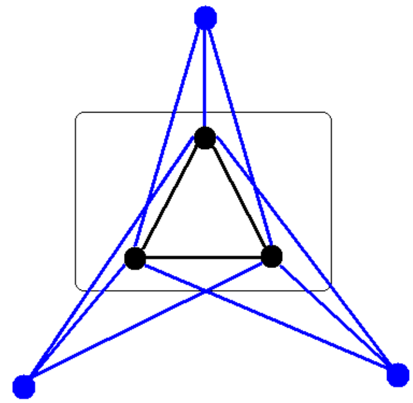

Figure 4: Example of a multistar with $|N| = |V'| = 3$

where $C \subset N$ is the *connector* vertex set.

Other inequalities are the so called *Generalized Large Multistar* (GLM) inequalities, discovered independently by several authors [17, 16]. GLMs can be written for $C-2LEV$ as

$$w\left(\delta(N)\right) \geq 2\sum_{j\in N}\frac{d_j}{K^2} + 2\sum_{j\in V_c/N}\frac{d_j}{K^2} w\left(E\left(N:\{j\}\right)\right) \quad \forall N \subseteq V_c \tag{40}$$

and they can be rewritten, according to demand conditions of customer sets, as

$$\sum_{k\in V_s}\sum_{\substack{i\in V' \\ j\in V_s\cup V_c\setminus N}} y_{ij}^k + \sum_{k\in V_s}\sum_{\substack{i\in V' \\ j\in V_s\cup V_c\setminus N}} y_{ji}^k \geq \tfrac{2}{K^2}\left(\sum_{\substack{i\in \\ j\in V_c\setminus N}} d_j y_{ij}^k + \sum_{\substack{i\in V_c\setminus S \\ j\in N}} d_i y_{ij}^k\right) + \\ +2\left\lceil \tfrac{d(N)}{K^2}\right\rceil \quad \forall N \subseteq V_c. \tag{41}$$

The heuristic separation procedure we use for checking HMs inequalities, partial multistar and *large multistar* class, which includes GLM inequalities, is the *polygon procedure*. Given a set of candidates, made through a selection based on support graph iteratively processed, checking potential violations is made by calculating an estimation on the maximum value of $w\left(E(N)\right)$ variables set in a feasible solution $UB(\alpha)$ varying $\alpha = w\left(E(N:V')\right)$. Then, a set of valid multistar inequalities are produced by solving the *convex hull* problem, related to estimation found, in α, $UB(\alpha)$ space. The separation algorithm of multistar and GLM class for 2E-CVRP is derived from [20, 19], adapting it to our $C-2LEV$ representation and, from it, to the original $C-2ECVRP$.

2.2 New classes of valid inequalities

By model (2)-(23), new classes of valid inequalities can be derived by considering variables which describe flow properties of the freight dispatched through network. In particular, we focus on inequalities directly involving the flow variables Q^2_{ijk}.

2.2.1 Additional Flow Constraints

Given a solution of 2E-CVRP, the following inequalities hold:

$$d_j y^k_{ij} \leq Q^2_{ijk} \quad \forall i \in V_c \cup V_s, \forall j \in V_c, \forall k \in V_s. \tag{42}$$

Inequalities (42) force a non-zero value of variables Q^2_{ijk} when their related arc variables y^k_{ij} have a non-zero value in the continuous solution. The number of potential violations is equal to the cardinality size of the set of arcs of the second level network and the check can be done in polynomial time.

2.2.2 Node Feasibility Inequalities

From the flow variables Q^2_{ijk} point of view, the feasibility of a node j, restricted to a satellite k, is assured in a general way from constraints set in the basic formulation. The following constraints

$$\sum_{i \in V_c \cup V_s} Q^2_{ijk} - \sum_{l \in V_c \cup V_s} Q^2_{jlk} = d_j y_{kj} \quad \forall j \in V_c, k \in V_s \tag{43}$$

have the same meaning of (10), restricted to V_c nodes set. We can state that, for any integer solution, on triplets $i, j \in V_c \cup V_s,\ k \in V_c$ only one between y^k_{ij} and y^k_{ji} is in the integer solution, when the route is a 2nd-level route which does not serve only one customer. In the following, we refer to a 2nd-level route serving exactly one customer as *single-customer route*.

Thereby, if a continuous solution of $C - 2ECVRP$ problem contains both flow variables referred to a given edge (i, j), then the following inequalities hold:

$$Q^2_{ijk} - \sum_{\substack{m \in V_c \cup V_s \\ m \neq i}} Q^2_{jmk} \leq d_j y^k_{ij} \quad \forall i \in V_c \cup V_s, j \in V_c, \forall k \in V_s \tag{44}$$

$$\sum_{\substack{i \in V_c \cup V_s \\ i \neq m}} Q^2_{ijk} - Q^2_{jmk} \geq d_j y^k_{jm} \quad \forall j \in V_c, \forall m \in V_c, \forall k \in V_s. \tag{45}$$

The inequalities (44) and (45) describe the possible node infeasibility problem generated by target incoming arc and target outgoing arc when both flow variables are active, respectively.

A possible violation can be detected considering the y^k_{ij} Q^2_{ijk} variables values associated to arcs incident upon selected node. The size of the set of potential violations is equal to the total number of arcs with both incident vertices in V_c set. The computational cost for (44) and (45) exact separation is $O\left(|V_c|^3\right)$.

2.2.3 Flow reformulation of existing CVRP inequalities

In $C-2LEV$ there is no explicit set of constraints linking flow behavior of the network to the routing information described by variables y_{ij}^k. A further method to get a possible strengthening effect on a generic LP solution is the reformulation of known CVRP inequality classes, when allowed, to an equivalent description directly involving flow variables Q_{ijk}^2.

Different classes of inequalities can be reformulated, by the replacement of arc variables with associated flow variables. In the *Generalized Large Multistar* inequality case [17, 18], which heuristic separation algorithm presented in [19] returns the candidate set V' with the maximum violation detected, generic (41) inequality is partitioned into $|V_s|$ inequalities, each one related to a single satellite k. Given a candidate set $V' \in V_c$, by (10), (11), (33) and by some simple manipulations on (40) we obtain the following inequalities:

$$\begin{aligned}&\sum_{\substack{i\in V'\\ j\in V\setminus V'}} y_{ij}^k + \sum_{\substack{i\in V'\\ j\in V\setminus V'}} y_{ji}^k \geq \tfrac{2}{K^2}\left(\sum_{\substack{i\in V'\\ j\in V_c\setminus V'}} d_j y_{ij}^k + \sum_{\substack{i\in V_c\setminus V'\\ j\in V'}} d_i y_{ij}^k\right)\\ &+\tfrac{2}{K^2}\cdot\left(d\left(V'\right)-\left(\sum_{\substack{l\in V_s\\ l\neq k}}\sum_{\substack{i\in V'\\ j\in V\setminus V'}} Q_{jil}^2 - \sum_{\substack{l\in V_s\\ l\neq k}}\sum_{\substack{i\in V'\\ j\in V\setminus V'}} Q_{ijl}^2\right)\right), \quad \forall k\in V_s,\end{aligned} \tag{46}$$

which are valid for the 2E-CVRP.

2.2.4 Satellite-Customers Route Connectivity Inequalities

By route feasibility of a solution, it is possible to introduce several valid inequalities by considering arc activating variables y_{ij}^k related to arcs connecting satellite and customer nodes.

Given a customer subset $V' \in V_c$, let us consider its maximum number of variables y_{ij}^k connecting V' to V_s. This can be done by computing the minimum number of arcs requested to serve $V_c \setminus V'$ and replacing $r\left(V_c\setminus V'\right)$ with its bound $\left\lceil \frac{d(V_c\setminus V')}{K^2}\right\rceil$, obtaining the following inequality

$$\sum_{k\in V_s}\sum_{j\in V'}\left(y_{jk}^k + y_{kj}^k\right) \leq |V'| + m_2 - \left\lceil \frac{d\left(V_c\setminus V'\right)}{K^2}\right\rceil. \tag{47}$$

Inequality (47) is valid for 2E-CVRP, for any $V' \subset V_c$, $1 \leq |V'| \leq m_2$. Let us remark that, if $|V'| = 1$, then (47) is equivalent to the *dominance rules on single-customer route* presented in [27].

Similarly, it is possible to extend single-customer route considerations to potential routes serving only a restricted number of customers. In this way, considering a customer set $V' \in V_c$, such that

$$m_2 - \left\lceil \frac{d\left(\bar{V}'\right)}{K^2}\right\rceil = 0, \tag{48}$$

there is no feasible solution containing a route with only one customer $j \in V'$. Thus,

$$\sum_{k \in V_s} \sum_{j \in V'} \left(y_{jk}^k + y_{kj}^k \right) + \sum_{k \in V_s} y^k \left(E(V') \right) \leq |V'| \qquad (49)$$

is valid for 2E-CVRP.

Then, considering a single satellite and by (16), (17), and (18), inequalities (49) can be decomposed into $|V_s|$ valid inequalities

$$\sum_{j \in V'} \left(y_{jk}^k + y_{kj}^k \right) + y^k \left(E(V') \right) \leq \sum_{j \in V'} z_{kj} \quad \forall k \in V_s. \qquad (50)$$

It is easy to prove that (50) dominates (49), being (49) the surrogated version of (50).

A heuristic separation is performed by selecting nodes in V' such that any feasible solution cannot have a route made only by one customer $j \in V'$, i.e.

$$m_2 - \left\lceil \frac{d \left(V_c \backslash V' \right)}{K^2} \right\rceil = 0.$$

In our heuristic, we consider candidate sets V', such that $|V'| \leq m_2$. The separation algorithm used for detecting (48) and (50) works as follows:

1. (a) Select from the graph given by the continuous relaxation solution candidate sets V' as the nodes such that $\sum_{k \in V_s} \left(y_{jk}^k + y_{kj}^k \right) > 1$;
   (b) Sort the nodes in V' by non-increasing demand;
   (c) Consider the node $j \in V'$ such that $m_2 - \left\lceil \frac{d(j)}{K^2} \right\rceil > 0$.
   (d) Remove j and check the potential violation of (48)

$$\sum_{k \in V_s} \sum_{j \in V'} \left(y_{jk}^k + y_{kj}^k \right) \leq |V'| + m_2 - \left\lceil \frac{d \left(V_c \backslash V' \right)}{K^2} \right\rceil .$$

2. (a) Build a candidate list V'' with nodes $i \in V_c$ such that $y_{ik}^k + y_{ki}^k > 0, \forall k \in V_s$.
   (b) For each couple $(i, j), i, j \in V''$, $i \neq j$, and for each satellite k, the violations of (50) are checked only if it exists, at least, one arc y_{ij}^k or y_{ji}^k in the continuous solution;
   (c) Set $V''' = \{i, j\}$ and add iteratively nodes $m \in V_c \backslash V'''$ such that does exists an arc between m and any node in V'''. Check (50) for all $k \in V_s$ after each addition and add the valid inequality when violated. Stop the process when $|V'''| = m_2$.

3 Computational Results

This section is devoted to present the computational results of the valid inequalities presented in the paper. In order to give a better insight of the 2E-CVRP and the valid inequalities themselves, they have been inserted in a Branch-and-Cut framework. The Branch-and-Cut has been implemented in SYMPHONY [26] interfaced with XPress 2008 [13]. As in [25], tests have been performed using a personal computer Intel 3.0 GHz with 1 GB RAM where a time limit of 10000 seconds has been imposed on each instance.

This section starts with a description in Subsection 3.1 of the considered instances and the separation sequences tested. Summary results and a brief analysis on the behavior of each sequence are presented in Subsection 3.2. Finally, in Subsection 3.3 general results are reported, using the best separation sequence for each problem.

3.1 Instance sets

For the tests, in order to compare the results with the literature we use the same instances of [25] and [12]. The instances cover up to 50 customers and 5 satellites and are grouped in two sets:

- Set 2 from [25]. The set contains 21 instances obtained as extensions of data sets E-n22-k4, E-n33-k4 and E-n51-k5 for the CVRP problem introduced in [8]. The cost matrix of each instance is given by the corresponding CVRP instance. The capacity of the 1st-level vehicles is 2.5 times the capacity of the 2nd-level vehicles, to represent cases in which the 1st-level is made by trucks and the 2nd-level is made by smaller vehicles (e.g., vehicles with a maximum weight less than 3.5 t). The capacity and the number of the 2nd-level vehicles is equal to the capacity of the vehicles of the VRP instance. The satellites are located at the same position of some randomly chosen customers. The instances range between 21 and 50 customers and consider 2 or 4 satellites.
- Set 4 from [12]. The main issue in the original instances by Christofides and Eilon is that the depot is in an almost baricentric position with respect to the customers. The instances of Set 4 have been generated with the depot out of the customers areas. Moreover, in order to represent different scenarios in city logistics, Crainic et al. [12] present 3 realistic distributions for the customers and 3 different strategies for the location of the satellites. Capacities for the global 2nd-level fleet and for each satellite are given. We consider 9 instances with 50 customers and 5 satellites.

All the instances are available in the OR-Library [7].

3.2 Analysis of the Valid Inequalities sequences

Many versions of the separation phase have been tried out to determine the best sequence. A separation sequence is defined by the order in which the inequality families are separated. In the following, we will refer to the different cut families as follows:

- A: TSP-based inequalities;
- B: Relaxed Capacity Inequalities;
- C: Node Feasibility Inequalities;
- D: Generalized Large Multistar Inequalities - Network flow formulation;
- E: Additional Flow Constraints;

- F: Strengthened Comb;
- G: Satellite-Customers Route Connectivity Inequalities.

In all the tables, we refer to the best results taken from the literature with the name *SOA* (State-Of-Art), while with the name *BC* we denote the results of our Branch-and-Cut algorithm. Tables 2 and 3 present results for the most interesting sequences of violated inequalities for instances of Set 2 and Set 4, respectively. Columns 1 and 2 give the instance name and the number of satellites of the instance. Column 3 reports the value of the lower bound obtained by continuous relaxation of model (2)-(23). The remaining columns report, for a given separation sequence, the value of the relaxation, the time needed for separating the cuts in seconds, the number of cuts and the percentage improvement from the lower bound of Column 3.

According to the results, inequalities of type F and G are quite time consuming, but help a lot to strengthen the formulation, in particular when the number of satellites increases. Furthermore, our tests show that a large amount of this computational time is due to the (2)-(23) model reoptimization and the simplex iterations involved. On the other hand, the lower bound at the root node is increased of about 5% in the mean both in Set 2 and Set 4. In particular, we note how the computational time spent in computing Satellite-Customers Route Connectivity Inequalities (G cuts) is the largest one, but it introduces a large improvement in the lower bound (see results in Table 3). Moreover, note that while the number of satellites increases, the number of violated Generalized Large Multistar Inequalities (D cuts) progressively reduces up to 0 (see results in Table 3).

Given the above results, in our Branch-and-Cut we adopted as final setting the separation $A+B+C+D+E+F+G$ at the root node. For the other nodes, due to the computational effort in the largest instances, we only apply the Lifted cover cuts and Lift-and-Project separation implemented in the XPress 2008 package [13]. The branching is prioritized on z_{jk} variables and a pseudocost-based rule for choosing the variable on which to branch is enabled on this variables.

3.3 General results and comparison with the literature

In this section we present the results of the full Branch-and-Cut and their comparison with the methods in literature [25]. The results of Set 2 and 4 are summarized in Tables 4 and 5, respectively. Columns 1 and 2 report the instance details, i.e. instance name and number of satellites in the instance. The *SOA* columns reports the best results taken from the literature, while columns *BC* show the behavior of our Branch-and-Cut. For each method, we give the best solution, the lower bound at the end of the optimization and the gap between them. Optimal values are reported in bold. The computational time is not reported, being fixed a priori to 10000 seconds for both methods.

From the results, we can notice how the Branch-and-Cut overcomes the cuts and the model introduced in [25] both in accuracy and number of optimal solutions. Our Branch-and-Cut is able to solve to optimality all the instances in Set 2 with 32 customers and one of the instances with 50 customers, for a total of 7 new instances solved to optimality. The

Instance	Satellites	SOA	A				A + B			
		LB	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %
E-n22-k4-s6-17	2	399.20	399.20	1	0	0.00%	399.20	1	47	0.00%
E-n22-k4-s8-14	2	358.24	358.24	1	0	0.00%	358.24	1	18	0.00%
E-n22-k4-s9-19	2	423.48	423.48	1	0	0.00%	423.48	1	45	0.00%
E-n22-k4-s10-14	2	348.22	348.22	1	0	0.00%	348.25	1	37	0.01%
E-n22-k4-s11-12	2	374.11	375.26	1	3	0.31%	376.48	2	49	0.63%
E-n22-k4-s12-16	2	349.70	351.17	1	2	0.42%	351.21	2	62	0.43%
E-n33-k4-s1-9	2	626.48	626.60	3	4	0.02%	626.60	3	51	0.02%
E-n33-k4-s2-13	2	610.21	611.15	3	4	0.15%	611.51	4	67	0.21%
E-n33-k4-s3-17	2	611.44	614.15	4	4	0.44%	614.75	5	100	0.54%
E-n33-k4-s4-5	2	636.93	646.89	3	3	1.56%	647.03	4	68	1.59%
E-n33-k4-s7-25	2	648.41	651.83	3	2	0.53%	652.10	4	80	0.57%
E-n33-k4-s14-22	2	662.62	663.84	3	1	0.18%	664.15	3	49	0.23%
E-n51-k5-s2-17	2	536.23	536.71	16	3	0.09%	536.71	20	132	0.09%
E-n51-k5-s4-46	2	502.85	506.01	20	2	0.63%	506.07	30	111	0.64%
E-n51-k5-s6-12	2	505.31	506.28	20	3	0.19%	506.28	24	122	0.19%
E-n51-k5-s11-19	2	544.35	544.67	18	0	0.06%	544.67	22	97	0.06%
E-n51-k5-s27-47	2	499.29	499.74	27	5	0.09%	499.74	32	142	0.09%
E-n51-k5-s32-37	2	513.01	513.91	19	3	0.17%	514.26	26	163	0.24%
E-n51-k5-s2-4-17-46	4	465.35	497.97	104	11	7.01%	498.10	129	263	7.04%
E-n51-k5-s6-12-32-37	4	462.99	498.49	124	10	7.67%	498.58	208	331	7.69%
E-n51-k5-s11-19-27-47	4	476.98	498.50	98	9	4.51%	498.52	114	203	4.52%

Instance	Satellites	SOA	A + B + C				A + B + C + D			
		LB	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %
E-n22-k4-s6-17	2	399.20	406.60	3	541	1.85%	406.60	3	541	1.85%
E-n22-k4-s8-14	2	358.24	366.81	3	596	2.39%	366.81	3	596	2.39%
E-n22-k4-s9-19	2	423.48	438.62	6	398	3.58%	438.62	3	398	3.58%
E-n22-k4-s10-14	2	348.22	358.17	3	389	2.86%	358.17	3	389	2.86%
E-n22-k4-s11-12	2	374.11	382.64	4	381	2.28%	387.28	5	391	3.52%
E-n22-k4-s12-16	2	349.70	366.44	4	565	4.79%	366.48	3	567	4.80%
E-n33-k4-s1-9	2	626.48	635.25	8	685	1.40%	635.25	7	685	1.40%
E-n33-k4-s2-13	2	610.21	619.90	12	1046	1.59%	619.90	9	1046	1.59%
E-n33-k4-s3-17	2	611.44	618.65	9	684	1.18%	618.65	8	684	1.18%
E-n33-k4-s4-5	2	636.93	653.98	9	568	2.68%	653.98	5	568	2.68%
E-n33-k4-s7-25	2	648.41	655.50	6	304	1.09%	655.50	4	304	1.09%
E-n33-k4-s14-22	2	662.62	669.55	5	258	1.05%	669.55	4	258	1.05%
E-n51-k5-s2-17	2	536.23	539.65	44	565	0.64%	539.65	34	565	0.64%
E-n51-k5-s4-46	2	502.85	508.27	61	1123	1.08%	508.27	38	1123	1.08%
E-n51-k5-s6-12	2	505.31	508.17	74	1547	0.57%	508.17	55	1547	0.57%
E-n51-k5-s11-19	2	544.35	547.95	41	1005	0.66%	547.95	30	1005	0.66%
E-n51-k5-s27-47	2	499.29	505.60	58	1251	1.26%	505.60	63	1251	1.26%
E-n51-k5-s32-37	2	513.01	516.54	53	1000	0.69%	516.54	40	1000	0.69%
E-n51-k5-s2-4-17-46	4	465.35	501.20	299	2003	7.70%	501.20	212	2003	7.70%
E-n51-k5-s6-12-32-37	4	462.99	501.77	421	2120	8.38%	501.77	331	2120	8.38%
E-n51-k5-s11-19-27-47	4	476.98	504.30	254	1849	5.73%	504.30	221	1849	5.73%

Instance	Satellites	SOA	A + B + C + D + E				A + B + C + D + E + F				A + B + C + D + E + F + G			
		LB	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %
E-n22-k4-s6-17	2	399.20	407.35	4	562	2.04%	408.04	5	546	2.21%	408.19	7	582	2.25%
E-n22-k4-s8-14	2	358.24	367.55	3	595	2.60%	368.24	5	615	2.79%	375.22	7	660	4.74%
E-n22-k4-s9-19	2	423.48	445.07	4	467	5.10%	441.37	8	449	4.23%	449.22	7	419	6.08%
E-n22-k4-s10-14	2	348.22	360.37	3	405	3.49%	360.24	9	418	3.45%	368.88	20	508	5.93%
E-n22-k4-s11-12	2	374.11	392.38	6	426	4.88%	393.46	6	438	5.17%	399.21	8	481	6.71%
E-n22-k4-s12-16	2	349.70	368.15	5	646	5.27%	368.98	7	699	5.51%	380.24	8	623	8.73%
E-n33-k4-s1-9	2	626.48	635.29	7	693	1.41%	636.95	20	753	1.67%	644.71	36	1107	2.91%
E-n33-k4-s2-13	2	610.21	620.13	12	1282	1.63%	625.08	28	1166	2.44%	645.47	46	1638	5.78%
E-n33-k4-s3-17	2	611.44	618.99	10	717	1.24%	626.98	35	1043	2.54%	642.16	38	993	5.03%
E-n33-k4-s4-5	2	636.93	653.98	9	568	2.68%	657.17	20	669	3.18%	677.25	38	753	6.33%
E-n33-k4-s7-25	2	648.41	655.43	5	302	1.08%	658.74	21	354	1.59%	663.62	38	479	2.35%
E-n33-k4-s14-22	2	662.62	669.58	5	261	1.05%	679.84	59	1252	2.60%	699.30	66	927	5.54%
E-n51-k5-s2-17	2	536.23	539.65	47	565	0.64%	539.87	104	607	0.68%	546.37	277	680	1.89%
E-n51-k5-s4-46	2	502.85	508.27	51	1111	1.08%	512.30	129	958	1.88%	517.83	534	1083	2.98%
E-n51-k5-s6-12	2	505.31	508.17	55	1547	0.57%	510.58	163	1802	1.04%	519.39	402	1731	2.79%
E-n51-k5-s11-19	2	544.35	547.95	44	1005	0.66%	549.31	93	1045	0.91%	556.77	282	1922	2.28%
E-n51-k5-s27-47	2	499.29	505.60	60	1251	1.26%	506.76	103	1193	1.49%	509.77	271	1487	2.10%
E-n51-k5-s32-37	2	513.01	516.66	50	1011	0.71%	516.96	118	1012	0.77%	524.98	354	1284	2.33%
E-n51-k5-s2-4-17-46	4	465.35	501.20	247	1978	7.70%	503.10	388	2098	8.11%	509.19	1671	2924	9.42%
E-n51-k5-s6-12-32-37	4	462.99	501.77	372	2120	8.38%	504.91	822	2163	9.05%	509.68	1704	2700	10.09%
E-n51-k5-s11-19-27-47	4	476.98	504.30	319	1849	5.73%	505.62	505	2428	6.01%	508.41	1075	2191	6.59%

Table 2: Analysis of different separation sequences - results on instances of Set 2

Instance	Satellites	SOA	A				A + B			
		LB	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %
37	5	1259.57	1261.43	83.00	11	0.15%	1261.56	100.00	281	0.16%
39	5	1239.74	1239.85	59.00	4	0.01%	1239.85	67.00	210	0.01%
41	5	1356.86	1361.29	88.00	13	0.33%	1361.43	120.00	261	0.34%
43	5	1124.17	1128.82	46.00	8	0.41%	1129.37	55.00	166	0.46%
45	5	1118.29	1119.83	40.00	7	0.14%	1119.83	44.00	156	0.14%
47	5	1230.82	1233.04	47.00	8	0.18%	1233.04	49.00	108	0.18%
49	5	1196.19	1200.37	68.00	17	0.35%	1201.28	78.00	270	0.43%
51	5	1116.96	1121.71	47.00	13	0.43%	1121.71	50.00	176	0.43%
53	5	1240.21	1240.99	57.00	14	0.06%	1245.06	75.00	229	0.39%

Instance	Satellites	SOA	A + B + C				A + B + C + D			
		LB	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %
37	5	1259.57	1286.56	233.00	2129	2.14%	1286.56	226.00	2129	2.14%
39	5	1239.74	1265.82	200.00	2500	2.10%	1265.82	191.00	2500	2.10%
41	5	1356.86	1385.44	225.00	2861	2.11%	1385.44	214.00	2861	2.11%
43	5	1124.17	1135.87	120.00	1089	1.04%	1135.87	116.00	1089	1.04%
45	5	1118.29	1124.16	73.00	919	0.53%	1124.16	70.00	919	0.53%
47	5	1230.82	1240.64	88.00	1063	0.80%	1240.64	83.00	1063	0.80%
49	5	1196.19	1212.95	172.00	2301	1.40%	1212.95	160.00	2301	1.40%
51	5	1116.96	1135.72	115.00	1915	1.68%	1135.72	106.00	1915	1.68%
53	5	1240.21	1251.84	159.00	2500	0.94%	1251.84	149.00	2500	0.94%

Instance	Satellites	SOA	A + B + C + D + E				A + B + C + D + E + F				A + B + C + D + E + F + G			
		LB	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %	LB	Time	Cuts	Improv. %
37	5	1259.57	1286.56	245.00	2129	2.14%	1291.59	440.00	2500	2.54%	1304.24	1142.00	2500	3.55%
39	5	1239.74	1265.82	209.00	2500	2.10%	1269.14	394.00	2500	2.37%	1288.65	1095.00	2500	3.95%
41	5	1356.86	1385.49	234.00	2859	2.11%	1399.04	450.00	2500	3.11%	1412.27	1128.00	2500	4.08%
43	5	1124.17	1136.05	132.00	1086	1.06%	1146.60	367.00	2500	1.99%	1223.96	1237.00	3000	8.88%
45	5	1118.29	1124.15	85.00	922	0.52%	1140.60	354.00	2279	2.00%	1194.42	1913.00	3192	6.81%
47	5	1230.82	1241.17	107.00	1082	0.84%	1257.60	368.00	1553	2.18%	1310.54	928.00	2109	6.48%
49	5	1196.19	1212.95	178.00	2301	1.40%	1220.63	396.00	2500	2.04%	1240.50	1039.00	2500	3.70%
51	5	1116.96	1135.73	122.00	1784	1.68%	1145.11	308.00	2099	2.52%	1170.70	1038.00	2500	4.81%
53	5	1240.21	1251.84	166.00	2500	0.94%	1273.45	1049.00	2926	2.68%	1306.02	1139.00	3000	5.31%

Table 3: Analysis of different separation sequences - results on instances of Set 4

		SOA			BC		
Instance	Satellites	Final Solution	Best Bound	Gap	Final Solution	Best Bound	Gap
E-n22-k4-s6-17	2	**417.07**	**417.07**	0.00%	**417.07**	**417.07**	0.00%
E-n22-k4-s8-14	2	**384.96**	**384.96**	0.00%	**384.96**	**384.96**	0.00%
E-n22-k4-s9-19	2	**470.60**	**470.60**	0.00%	**470.60**	**470.60**	0.00%
E-n22-k4-s10-14	2	**371.50**	**371.50**	0.00%	**371.50**	**371.50**	0.00%
E-n22-k4-s11-12	2	**427.22**	**427.22**	0.00%	**427.22**	**427.22**	0.00%
E-n22-k4-s12-16	2	**392.78**	**392.78**	0.00%	**392.78**	**392.78**	0.00%
E-n33-k4-s1-9	2	730.16	725.50	0.64%	**730.16**	**730.16**	0.00%
E-n33-k4-s2-13	2	714.63	701.04	1.94%	**714.63**	**714.63**	0.00%
E-n33-k4-s3-17	2	707.62	683.42	3.54%	**707.41**	**707.41**	0.00%
E-n33-k4-s4-5	2	787.29	764.80	2.94%	**778.73**	**778.73**	0.00%
E-n33-k4-s7-25	2	766.49	739.24	3.69%	**756.84**	**756.84**	0.00%
E-n33-k4-s14-22	2	779.19	764.38	1.94%	**779.05**	**779.05**	0.00%
E-n51-k5-s2-17	2	599.20	576.97	3.85%	597.51	556.55	7.36%
E-n51-k5-s4-46	2	561.80	513.09	9.49%	530.76	529.34	0.27%
E-n51-k5-s6-12	2	593.71	526.91	12.68%	554.80	541.17	2.52%
E-n51-k5-s11-19	2	646.66	550.99	17.36%	584.09	558.27	4.63%
E-n51-k5-s27-47	2	538.22	524.00	2.71%	538.22	535.04	0.59%
E-n51-k5-s32-37	2	553.64	540.14	2.50%	**552.27**	**552.27**	0.00%
E-n51-k5-s2-4-17-46	4	694.83	502.82	38.19%	542.37	515.75	5.16%
E-n51-k5-s6-12-32-37	4	571.80	509.35	12.26%	539.02	516.02	4.46%
E-n51-k5-s11-19-27-47	4	724.09	506.99	42.82%	589.14	511.09	15.27%
Mean				8.70%			1.92%

Table 4: Comparison with State-of-the-art on Set 2

trend is confirmed by the mean gap, which is less than 2%. In particular, notice how the gap is reduced in the instances with 4 satellites, which where the most problematic in literature.

The same behavior is confirmed also in Set 4, where the mean gap is reduced by more than 7 percentage points. Finally, notice how the mean of Set 4, which considers instances with 5 satellites, is similar to the mean of instances with 4 satellites in Set 2, which is about 8%, prooving that, even increasing the number of satellites, the mean gap obtained by the Branch-and-Cut is stable. Moreover, the reduction of the gap from 16.75% to 9.35% in Set 4 is due only to the valid inequalities. In fact, the best integer solutions obtained in literature and by the Branch-and-Cut are the same.

In the meantime, the results show also some limits of the MIP model, which heavily affect the global computational times.

		SOA			BC		
Instance	Satellites	Final Solution	Best Bound	Gap	Final Solution	Best Bound	Gap
Instance50-s5-37.dat	5	1548.07	1355.8	14.18%	1548.07	1434.54	7.91%
Instance50-s5-39.dat	5	1525.24	1365.78	11.68%	1525.24	1423.48	7.15%
Instance50-s5-41.dat	5	1775.06	1453.03	22.16%	1775.06	1580.8	12.29%
Instance50-s5-43.dat	5	1455.4	1259.55	15.55%	1455.4	1341.04	8.53%
Instance50-s5-45.dat	5	1497.91	1246.26	20.19%	1497.91	1331.59	12.49%
Instance50-s5-47.dat	5	1621.48	1405.34	15.38%	1621.48	1487.09	9.04%
Instance50-s5-49.dat	5	1499.52	1297.08	15.61%	1499.52	1370.83	9.39%
Instance50-s5-51.dat	5	1436.3	1265.66	13.48%	1436.3	1289.45	11.39%
Instance50-s5-53.dat	5	1571.34	1385.72	13.40%	1571.34	1483.19	5.94%
Mean				16.75%			9.35%

Table 5: Comparison with State-of-the-art on Set 4

4 Conclusions

In this work we considered 2E-CVRP, a recently introduced multi-echelon extension of the classical CVRP, and derived several classes of valid inequalities in order to strengthen its formulation. We both extended classical valid inequalities from TSP and VRP problems and derived specific families of valid inequalities for 2E-CVRP.

The proposed Branch-and-Cut has been tested on benchmarks taken from the literature and covering instances with up 50 customers and 5 satellites.

Our results show that the new inequalities are effective, being able to solve to optimality 7 new instances in one set of instances and reduce by almost 50% the gap in Set 4. Moreover, inequalities which are usually effective on the CVRP, the Generalized Large Multistar Inequalities, are not able to tackle instances where the number of the satellites is increasing.

Future research will be aimed to create new formulations able to manage larger problems, as well as to exploit set-partitioning based formulations of the problem.

5 Acknowledgments

This project has been partially supported by the Ministero dell'Università e della Ricerca (MIUR) (Italian Ministry of Education, University and Research), under the Progetto di Ricerca di Interesse Nazionale (PRIN) 2007 "Optimization of Distribution Logistics".

Funding has also been provided by the the Natural Sciences and Engineering Council of Canada (NSERC), through its Industrial Research Chair and Discovery Grants programs, by the partners of the Chair, CN, Rona, Alimentation Couche-Tard and the Ministry of Transportation of Québec, and by the Fonds québécois de recherche sur la nature et les technologies (FQRNT Québec) through its Team Research grants program.

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