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A Multi-Stage Stochastic Programming Approach for Production Planning with Uncertainty in the Quality of Raw Materials and Demand

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Abstract. Motivated by the challenges encountered in sawmill production planning, we study a multi-product, multi-period production planning problem with uncertainty in the quality of raw materials and consequently in processes yields, as well as uncertainty in products demands. As demand and yield own different uncertain natures, they are modeled separately and then integrated. Demand uncertainty is considered as a dynamic stochastic data process during the planning horizon which is modeled as a scenario tree. Each stage in demand scenario tree corresponds to a cluster of time periods, for which the demand has a stationary behavior. The uncertain yield is modeled as scenarios with a stationary probability distribution during the planning horizon. Yield scenarios are then integrated in each node of demand scenario tree, constituting a hybrid scenario tree. Based on the hybrid scenario tree for the uncertain yield and demand, a multi-stage stochastic programming (MSP) model is proposed which is full recourse for demand scenarios and simple recourse for yield scenarios. We conduct a case study with respect to a realistic scale sawmill. Numerical results indicate that the solution to the multi-stage model is far superior to the optimal solution to the mean-value deterministic and the two-stage stochastic models.

Keywords. Production planning, random yield, random demand, sawmill, scenario tree, multi-stage stochastic programming.

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1. Introduction

Production planning is a key area of operations management. The plans have to be determined in the face of uncertainty in environmental and system uncertainties, namely uncertain products demands, processes yields, etc. An important methodology for production planning is mathematical programming. Traditional mathematical programming models for production planning are deterministic, and may result unsatisfactory production plans in the presence of uncertainty.

The goal of this work is to address a multi-period, multi-product (MPMP) production planning problem in a manufacturing environment where alternative processes produces simultaneously multiple products from several classes of raw materials. Besides, raw materials own non-homogeneous and random characteristics (e.g. logs in sawmills, or crud oil in refineries). Thus, the quantities of products that can be produced by each process (processes yields) are random variables. Moreover, market demand for products is also uncertain and non-stationary during the planning horizon. The production planning problem we are studying includes deciding how many times each process should be run and which quantity of each class of raw materials should be consumed by each process in each period in the planning horizon. The objective is to minimize products inventory/backorder and raw material consumption costs, regarding fulfillment of products demands, machine capacities, and raw material inventory. This work is motivated by production planning for sawing units in sawmills, where the processes yields are random variables due to non-homogeneity in the characteristics of logs, and lumber demand is also uncertain.

A review of some of the existing literature of production planning under uncertainty is provided in Mula et al. (2006). Stochastic programming (Dantzig, 1955; Kall and Wallace, 1994; Birge and Louveux 1997; Kall and Mayer, 2005) and robust optimization (Mulvey et al., 1995) has seen several successful applications in production planning. In Escudero et al. (1993) a multi-stage stochastic programming approach was used for addressing a MPMP production planning model with random demand. Bakir and Byrune (1998) developed a stochastic LP model based on the two-stage deterministic equivalent problem to incorporate demand uncertainty in a multi-period multi-product (MPMP) production planning model. Huang K. (2005) proposed multi-stage stochastic programming models for production and capacity planning under uncertainty. Alfieri and Brandimarte (2005) reviewed multi-stage stochastic models applied in multi-period production and capacity planning in the manufacturing systems. Brandimarte (2006) proposed a multi-stage programming approach for multi-item capacitated lot-sizing with uncertain demand. Kazemi et al. (2007) proposed a two-stage stochastic model for addressing MPMP production planning with uncertain yield. Khor et al. (2007) proposed a two-stage stochastic programming model as well as robust optimization models for capacity expansion planning in petroleum refinery under uncertainty. Leung and Wu. (2004) proposed a robust optimization model for stochastic aggregate production planning. In Leung et al. (2006) a robust optimization model was developed to address a multi-site aggregate production planning problem in an uncertain environment. Wu (2006) applied the robust optimization approach to uncertain production loading problems with import quota limits under the global supply chain management environment. Kazemi et al. (2008b) proposed two robust optimization models with different recourse cost variability measures to address MPMP production planning with uncertain yield.

Adopting a two-stage approach in the uncertain multi-period production planning literature (see e.g. Bakir et al., 1998; Kazemi et al., 2007, 2008a, b; Khor et al., 2007) cannot model the dynamic decision process in such problems. In a two-stage approach, the plan for the entire multi-period planning horizon is determined before the uncertainty is realized, and only a limited number of recourse actions can be taken afterwards. In contrast, a multi-stage approach allows revision of the planning decisions as more information regarding the uncertainties is revealed. Consequently, the multi-stage model is a better characterization of the dynamic planning process, and provides more flexibility than does the two-stage model.

In the existing contributions in the literature for production planning with uncertainty, either one uncertain parameter (e.g. either demand or yield) is taken into account (Escudero et al., 1993; Bakir et al., 1998; Brandimarte, 2006; Kazemi et al., 2007, 2008a, b) or one set of scenarios or a scenario tree is considered for all the uncertain parameters simultaneously (Leung and Wu., 2004; Huang K., 2005; Leung et al. 2006; Wu, 2006; Khor et al., 2007). However, when the uncertain parameters own different dynamics and behavior over time and each might need different sorts of recourse actions, it is more realistic to model them separately and then integrate them to be used in the stochastic programming models.

In this paper, we propose a multi-stage stochastic program for MPMP production planning with uncertain yield and demand. As demand uncertainty originates from market conditions and yield uncertainty is due to non-homogeneity in the quality of raw materials, they are modeled separately and independently. We assume that the uncertain demand evolves as a discrete time stochastic process during the planning horizon with a finite support. This information structure can be interpreted as a scenario tree. Each stage in demand scenario tree corresponds to a cluster of time periods. It is supposed that demand has a stationary behavior during the periods at each stage. The uncertain yields are modeled as scenarios with stationary probability distribution during the planning horizon. Finally, yield scenarios are integrated into the demand scenario tree, forming a hybrid scenario tree with two types of branches in each node. Depending on the availability of information on the uncertain parameters at the beginning of each stage in the scenario tree, different recourse actions are defined for them in the multi-stage stochastic model. We suppose that at the beginning of each stage in demand scenario tree, the decision maker has a perfect insight on the demand scenario that will be observed at that stage. Thus, the production plan can be adjusted for demand scenarios (full recourse). On the other hand, as yield scenarios are revealed after plan implementation, production plan is constant for yield scenarios (simple recourse). The goal of the multi-stage stochastic model is to determine implementable plans for production that takes into account the possible demand and yield scenarios, provide for recourse actions in the future, and minimize the expected cost of raw material consumption, holding inventory, and backorders. It should be noted that the multi-stage model was represented as compact formulation (see for example Alfieri and Brandimarte, 2005) based on the scenarios of the hybrid scenario tree, in order to have a deterministic equivalent model of manageable size that can be solved by CPLEX. The proposed approach is applied for sawmill production planning under the uncertainty in raw material (log) quality and product (lumber) demand. Regarding the large dimensionality of the resulted deterministic equivalent model for a realistic scale sawmill, the periods in the planning horizon are clustered into three stages. As a result, the original multi-stage model is approximated by a 4-stage one. Numerical results indicate that the solution to the multi-stage model is far superior to the optimal solution of the mean-value deterministic and the two-stage stochastic models. Furthermore, it is shown that the significance of using multi-stage stochastic programming is increased as the variability of random demand is augmented in the scenario tree.

The remainder of this paper is organized as follows. In the next section, a theoretical framework for multi-stage stochastic programming (MSP) is provided. In section 3, we provide a multi-stage stochastic linear program for MPMP production planning with random yield and demand. In section 4, we describe one of the applications of this problem which is sawmill production planning under the uncertainty in raw material (log) quality and product (lumber) demand. In section 5, the implementation results of the multi-stage stochastic model for a prototype realistic scale sawmill are presented. Our concluding remarks are given in section 6.

2. Multi-stage stochastic programming

In a problem where time and uncertainty play an important role, the decision model should be designed to allow the user to adopt a decision policy that can respond to events as they unfold. The specific form of the decisions depends on assumptions concerning the information that is available to the decision maker, when (in time) is it available and what adjustments (recourse) are available to the decision maker. Multi-stage stochastic programming (MSP) approach (Kall and Wallace, 1994; Birge, and Louveux 1997; Kall and Mayer, 2005) was proposed to address multi-period optimization models with dynamic stochastic data during the time. In multi-stage stochastic programming (MSP) a lot of emphasis is placed on the decision to be made today, given present resources, future uncertainties and possible recourse actions in the future. The uncertainty is represented through a scenario tree and an objective function is chosen to represent the risk associated with the sequence of decisions to be made and the whole problem is then solved as a large scale linear or quadratic program. In the following, we first review the characteristics of scenario trees, and then we provide a general formulation for multi-stage stochastic programming.

2.1. Scenario tree

Scenario tree is a computationally viable way of discretizing the underlying dynamic stochastic data over time in a problem. An illustration of scenario tree is provided in Figure 1. In a scenario tree, each stage denotes the stage of the time when new information is available to the decision maker. Thus, the stages do not necessarily correspond to time periods. They might include a number of periods in the planning horizon. Scenario tree consists of a number of nodes and arcs at each stage. Each node n in the scenario tree represents a possible state of the world, associated with a set of data (stochastic demand, stochastic cost, etc.) in the corresponding stage. The root node of the tree represents the current state of the world. The branches (arcs) in the scenario tree denote the scenarios for the next stage. A probability is associated to each arc of scenario tree which denotes the probability of the corresponding scenario to that arc. It should be noted that, the probability of each node in the scenario tree is computed as the product of probability of the arcs from the root node to that node. Furthermore, the sum of probabilities of nodes at each stage is equal to 1. A path from the root node to a node n describes one realization of the stochastic process from the present time to the period where node n appears. A full evolvement of the stochastic process over the entire planning horizon, i.e., the path from the root node to a leaf node, is called a scenario. In the scenario tree example of figure 1, we have 4 stages. Each node *n* in the tree has two branches to the next stage which denote two possible scenarios for the next stage, when we are at stage *n*. Consequently, we have 8 scenarios by the end of stage 4. A review of approaches for generating the scenario trees for multi-stage stochastic programs, based on the underlying random data process is provided in (Dupačová et al., 2000).

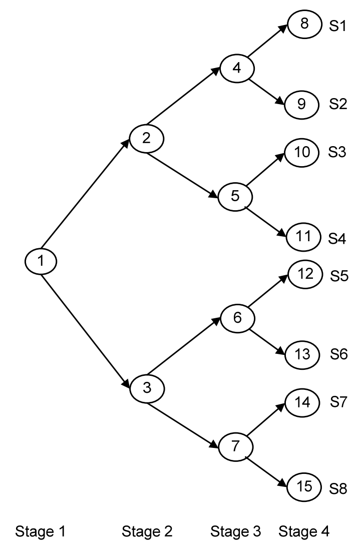

Figure 1. Scenario tree for multi-stage stochastic programming

2.2. Multi-stage stochastic programming models

We begin by abstracting the statement of a multi-period deterministic LP model:

$$\text{Minimize } c_1 x_1 + c_2 x_2 + \ldots + c_T x_T \qquad (1)$$

Subject to

$$\begin{aligned} &A_{11} x_1 &&= b_1 \\ &A_{21} x_1 + A_{22} x_2 &&= b_2 \\ &. \\ &. \\ &. \\ &A_{T1} x_1 + \ldots \quad + A_{TT} x_T &&= b_T \\ &x_1 \geq 0, x_2 \geq 0, \ldots, x_T \geq 0 \end{aligned}$$

Let the scenario *s* correspond to a single setting of all data in this problem,

$s = \{ c_t, b_t, A_{tt'} : t = 1, \ldots, T, t' = 1, \ldots, T \}$

and a decision *x* corresponds to a setting of all the decision variables

$x : (x_1, \ldots, x_T) \in R^{n_1} \times \ldots \times R^{n_T}$

Solving the deterministic LP model (1) for a given setting s of the data is equivalent to solving the following problem for a certain function:

$\min f(x,s) \qquad \text{over all } x,$

where

$$f(x,s)=\begin{cases}\sum_{t=1}^{T} c_t x_t, & \text{if } x \text{ satisfies all constraints in (1)},\\ +\infty, & \text{otherwise.}\end{cases}$$

We next develop the stochastic model. Let us suppose that we are given a set S of scenarios. The decision-maker wishes to set a policy that makes different decisions under different scenarios. Mathematically, a policy X that assigns to each scenario $s \in S$ is a vector $X(s) := (X_1(s),...,X_T(s))$, where $X_t(s)$ denotes the decision to be made at stage t if encountered by scenario s. Decisions that depend on the individual scenarios do not hedge against the possibility that the scenario may not occur, leaving one vulnerable to disastrous consequences if some other scenario does happen. In other words, our decision process must conform to the flow of available information, which basically means the decisions must be non-anticipative (or implementable). A decision is said to be implementable if for every pair of scenarios s and s' that are indistinguishable up to stage t

$(X_1(s),...,X_t(s)) = (X_1(s'),...,X_t(s'))$.

As examples of indistinguishable scenarios, we can refer to scenarios 1, 2, 3, 4 in node 2, at stage 2 of scenario tree in figure 1. Implementability guaranties that policies do not depend on information that is not yet available. The multi-stage stochastic programming can be formulated as:

$$\min\left\{\sum_{s\in S} p^s f(X(s),s) \mid X \text{ is an implementable policy}\right\},$$

where p^s denotes the probability of scenario s. There are two approaches to impose the non-anticipativity constraints in the multi-stage stochastic programs which lead to *split variable* formulation and *compact* formulation.

2.2.1. Split variable formulation

In *split variable* formulation, we introduce a set of decision variables for each stage and each scenario, and then we enforce non-anticipativity constraints explicitly based on the shape of scenario tree. Although this representation increases the problem dimensions, it yields a sparsity structure that is well suited to the interior point algorithms. Alternatively, it is possible to use a decomposition approach on the splitting variables formulation. Several strategies have been published in the literature for solving large-scale multi-stage stochastic programs (Ruszczynski 1989; Rockafellar and Wets 1991; Mulvey and Ruszczyński 1995; Liu and Sun 2004).

2.2.2. Compact formulation

In *compact* formulation, we associate decision variables to the nodes of scenario tree and build non-anticipativity in an implicit way. In other words, the variables such as $X(s)$ for $X(s) = X(s')$ are replaced in the model by one single variable, and redundant constraints for partially identical scenarios are deleted. Compact formulations are computationally cheaper when using for solving by the Simplex methodology in the standard solvers.

3. Model development

In this section, we first present a deterministic mathematical formulation for the problem under consideration. Then, we provide the multi-stage stochastic formulation to address the problem by considering the uncertain processes yields and products demands.

3.1. A deterministic model for multi-product, multi-period production planning

Consider a production unit with a set of products P, a set of classes of raw materials C, a set of production processes A, a set of machines R, and a planning horizon consisting of *T* periods. To state the deterministic linear programming model for this problem, the following notations are used:

3.1.1. Notations

<u>Indices</u>

p product

t period

c raw material class

a production process

r machine

<u>Parameters</u>

h_{pt} Inventory holding cost per unit of product p in period t

b_{pt} Backorder cost per unit of product p in period t

m_{ct} Raw material cost per unit of class c in period t

I_{c0} The inventory of raw material class c at the beginning of planning horizon

I_{p0} The inventory of product p at the beginning of planning horizon

s_{ct} The quantity of material of class c supplied at the beginning of period t

d_{pt} Demand of product p by the end of period t

ϕ_{ac} The units of class c raw material consumed by process a (consumption factor)

ρ_{ap} The units of product p produced by process a (yield of process a)

δ_{ar} The capacity consumption of machine r by process a

M_{rt} The capacity of machine r in period t

Decision variables

X_{at} The number of times each process a should be run in period t

I_{ct} Inventory size of raw material of class c by the end of period t

I_{pt} Inventory size of product p by the end of period t

B_{pt} Backorder size of product p by the end of period t

3.1.2. The deterministic LP model

$$\text{Minimize } Z = \sum_{p \in \mathrm{P}} \sum_{t=1}^{T} (h_{pt} I_{pt} + b_{pt} B_{pt}) + \sum_{c \in \mathrm{C}} \sum_{t=1}^{T} \sum_{a \in A} m_{ct} \phi_{ac} X_{at} \tag{2}$$

Subject to

$$I_{ct} = I_{ct-1} + s_{ct} - \sum_{a \in A} \phi_{ac} X_{at}, \qquad t = 1, \dots, T,\ c \in \mathrm{C}, \tag{3}$$

$$\begin{aligned} &I_{p1} - B_{p1} = I_{p0} + \sum_{a \in \mathrm{A}} \rho_{ap} X_{a1} - d_{p1}, \\ &I_{pt} - B_{pt} = I_{pt-1} - B_{pt-1} + \sum_{a \in \mathrm{A}} \rho_{ap} X_{at} - d_{pt}, \qquad t = 2, \dots, T,\ p \in \mathrm{P}, \end{aligned} \tag{4}$$

$$\sum_{a \in \mathrm{A}} \delta_{ar} X_{at} \le M_{rt}, \qquad t = 1, \dots, T,\ r \in \mathrm{R}, \tag{5}$$

$$X_{at} \ge 0, I_{ct} \ge 0, I_{pt} \ge 0, B_{pt} \ge 0, \qquad t = 1, \dots, T,\ p \in \mathrm{P},\ c \in \mathrm{C},\ a \in A. \tag{6}$$

The objective function (2) minimizes total inventory and backorder costs for all products and raw material cost for all classes in the planning horizon. Constraint (3) ensures that the total inventory of raw material of class c at the end of period t is equal to its inventory in the previous period plus the quantity of material of class c supplied at the beginning of that period (s_{ct}) minus its total consumption in that period. Constraint (4) ensures that the sum of inventory (or backorder) of product p at the end of period t is equal to its inventory (or backorder) in the previous period plus the total production of that product in that period, minus the product demand for that period. Total quantity of production for each product in each period is calculated as the sum of the quantities yielded by each of the corresponding processes, regarding the yield (ρ_{ap}) of each process. Finally, constraint (5) requires that the total production do not exceed the available production capacity.

3.2. Multi-stage stochastic programming extension

In this section, we first describe our proposed approach to model the uncertain yield and demand, and then provide the production planning formulation by multi-stage stochastic programming.

3.2.1. Modeling the uncertain yield and demand

We assume that the uncertain demand evolves as a discrete time stochastic process during the planning horizon with a finite support. This information structure can be interpreted as a scenario tree (see figure 1 in section 2). The nodes at stage t of the tree constitute the states (scenarios) of demand that can be distinguished by information available up to stage t. For each stage a limited number of demand scenarios are taken into account (e.g. *high*, *average*, *low*). In order to define the scenarios for each stage, we can either use the traditional approach of making distributional assumptions, estimating the parameters from historical data, or use the scenarios proposed by the experts. In order to keep the resulting multi-stage stochastic model within a manageable size, we assume that the planning horizon is clustered into N stages, where each stage includes a number of periods. In other words, it is supposed that the uncertain demand is stationary during the time periods at each stage. For example, if the demand scenario for the first period at stage n is *high*, it remains the same (*high*) for the rest of periods at stage n; however the demand scenario might change (e.g. to *low*) for the first period in the next stage ($n+1$). It should be noted that, the number of periods that can be considered at each stage depends on the behavior of demand in the industry, as well as the length of planning horizon.

On the other hand, we assume that raw materials are supplied from the same supply source during the planning horizon. Thus, it is supposed that the uncertain yield has stationary probability distribution.

The probability distribution of random yield is estimated based on historical data in industry. A number of scenarios are taken in to account for yields by discritization of the original probability distribution. Regarding to the stationary distribution of yield, only one of the scenarios can take place during the planning horizon.

In order to have a single stochastic production planning model that considers uncertain yield and demand, yield scenarios are integrated with the demand scenario tree forming a hybrid scenario tree. An example of a four-stage hybrid scenario tree is depicted in figure 2, where full line branches denote demand scenarios while dashed line branches denote yield scenarios. At each node of the tree, which denotes one demand scenario for the corresponding stage, different yield scenarios can take place (3 scenarios in the example of figure 2). However, regarding the stationary behavior of uncertain yield, only one of the yield scenarios can be observed during the planning horizon. Thus, the total number of scenarios in the hybrid scenario tree can be computed as the number of leaves in demand scenario tree by the number of yield scenarios (in the example of figure 2, this number is equal to 24).

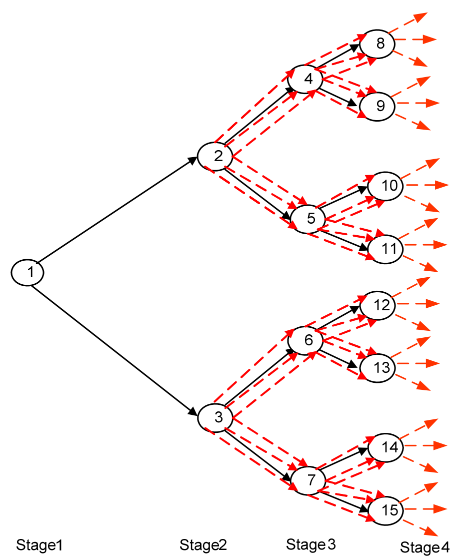

Figure 2. A hybrid scenario tree for uncertain demand and yield

3.2.2. Multi-stage stochastic program for MPMP production planning with uncertain yield and demand

Let us now formulate the problem as a multi-stage stochastic (MSP) model based on the hybrid scenario tree for the uncertain yield and demand. The decision (control) variables of deterministic model (2)-(6) are production plans X_{at}. The inventory and backorder variables I_{pt} and B_{pt} are the consequences (state variables) of the plan. In this problem, we assume that the decision maker can adjust the production plan X_{at} for different demand scenarios at each stage of demand scenario tree. In other words, it is supposed at the beginning of each stage, enough information on demand is available to the decision maker to select properly among the plans proposed by the MSP model for different scenarios. Thus we have a model with full recourse with respect to demand scenarios. As we use compact formulation to represent the problem, the decision variables X_{at} are defined for each node of demand scenario tree. On the other hand, as the quality of materials is not is not known before production, the yield scenarios can only be revealed after implementation of production plan. Thus, the production plan for each node of demand scenario tree should be fixed for all the yield scenarios. In other words, the model becomes simple recourse with respect to yield scenarios. It is evident that the inventory and backorder of products in each period ($I_{pt}^{i}(n)$ and $B_{pt}^{i}(n)$), which are the state variables, depend on the demand scenarios as well as yield scenarios, thus they are indexed for yield scenarios as well as demand nodes. Regarding the above discussions, the following notations in addition to those provided in 3.1.1 are used in the multi-stage model. The compact formulation of multi-stage model follows by the notations.

3.2.2.1. Notations

<u>Indices</u>

Tree Scenario tree.

S Number of scenarios for random yields.

i Scenario of random yield.

n,m Node of scenario tree.

$a(n)$ Ancestor of node n in the scenario tree.

t_n Set of time periods corresponding to node n in the scenario tree.

<u>Parameters</u>

$d_{pt}(n)$ Demand of product p by the end of period t at node n of the scenario tree.

$p(n)$ Probability of node n of the scenario tree.

p^i Probability of scenario i for random yield.

<u>Decision variables</u>

$X_{at}(n)$ The number of times each process a should be run in period t at node n of the scenario tree.

$I_{ct}(n)$ Inventory size of raw material of class c by the end of period t at node n of the scenario tree.

$I_{pt}^{i}(n)$ Inventory size of product p by the end of period t for scenario i of random yield at node n of the scenario tree.

$B^i_{pt}(n)$ Backorder size of product *p* by the end of period *t* for scenario *i* of random yield at node *n* of the scenario tree.

3.2.2.2. Multi-stage stochastic model (compact formulation)

$$\text{Minimize}\, Z = \sum_{n\in Tree} p(n)(\sum_{t\in t_n}\sum_{c\in C}\sum_{a\in A} m_{ct}\phi_{ac}X_{at}(n)) + \sum_{n\in Tree} p(n)(\sum_{i=1}^{S} p^i(\sum_{t\in t_n}\sum_{p\in P}(h_{pt}I^i_{pt}(n)+b_{pt}B^i_{pt}(n)))) \quad (7)$$

Subject to

$$I_{ct}(n) = I_{ct-1}(m) + s_{ct} - \sum_{a\in A}\phi_{ac}X_{at}(n), \quad n\in Tree,\, t\in t_n,\, c\in C,\quad m=\begin{cases} a(n), & t-1\notin t_n,\\ n, & t-1\in t_n,\end{cases} \quad (8)$$

$$\sum_{a\in A}\delta_{ar}X_{at}(n) \le M_{rt}, \quad n\in Tree,\, t\in t_n,\, r\in R, \quad (9)$$

$$I^i_{pt}(n) - B^i_{pt}(n) = I^i_{pt-1}(m) - B^i_{pt-1}(m) + \sum_{a\in A}\rho^i_{ap}X_{at}(n) - d_{pt}(n), \quad n\in Tree,\, t\in t_n,\, p\in P,\, i=1,\dots,S,\quad m=\begin{cases} a(n), & t-1\in t_n,\\ n, & t-1\notin t_n,\end{cases} \quad (10)$$

$$X_{at}(n)\ge 0,\, I_{ct}(n)\ge 0,\, I^i_{pt}(n)\ge 0,\, B^i_{pt}(n)\ge 0, \quad n\in Tree,\, t\in t_n,\, c\in C,\, p\in P,\, a\in A,\, i=1,\dots,S. \quad (11)$$

The first term of the objective function (7) accounts for the expected material cost for demand nodes of the scenario tree. The second term is the expected inventory and backorder costs for demand nodes and yield scenarios. In model (7)-(11), the decision variables are indexed for each node, as well as for each time period, since the stages do note correspond to time periods. As it was mentioned in 3.2.1, each node at a stage includes a set of periods which is denoted by t_n. In this model, there are coupling variables between different stages and these are the ending inventory and backorder variables at the end of each stage. As it can be observed in this model, two different node indices (*n*, *m*) are used for inventory/backorder variables in the inventory balanced constraints ((8) and (10)). More precisely, for the first period at each stage, the inventory or backorder is computed by considering the inventory or backorder of previous period corresponding to its ancestor node, while for the rest of periods in that stage, the inventory/backorder size of previous period corresponding to the same node are taken into account.

4. Case study: sawmill production planning

In this section, we introduce one of the applications of the general problem already described in this paper, which is sawmill production planning. There are a number of processes that occur at a sawmill: log sorting, sawing, drying, planing and grading (finishing). Raw materials in sawmills are the logs which are transported from different districts of forest after bucking the felled trees. The finished and graded lumbers (products) are then transported to the domestic and international markets. Figure 3 illustrates the typical processes. As a case study, we consider the sawing units in sawmills. In the sawing units, logs are classified according to some attributes namely: diameter class, species, length, taper, etc. Logs are broken down into different dimensions of lumbers by means of different cutting patterns. See figure 4 for three different cutting patterns. Each cutting pattern is a combination of activities that are run on a set of machines. From each log, several pieces of sawn lumber (e.g. $2(in)\times 4(in)\times 8(ft)$, $2(in)\times 4(in)\times 10(ft)$, $2(in)\times 6(in)\times 16(ft),\ldots$) are produced depending on the cutting pattern. The lumber quality (grade) as well as its quantity yielded by each cutting pattern depends on the quality and characteristics of the input logs. Despite the classification of logs in sawmills, variety of characteristics might be observed in different logs in each class. In fact, due to natural variable conditions that occur during the growth period of trees, non-homogeneous and random characteristics (in terms of diameter, number of knots, internal defects, etc.) can be observed in different logs in each class, make it impossible to anticipate the exact yield of a log.

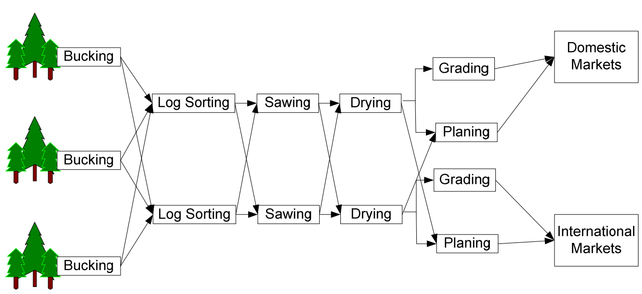

Figure 3.Illustration of sawmills processes

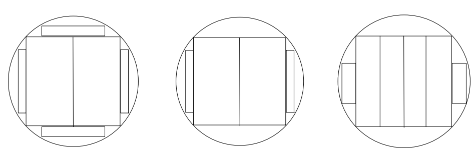

Figure 4. Three possible cutting patterns in a sawmill

As it is not possible in many sawmills to scan the logs before planning, the exact yields of cutting patterns for different log classes cannot be determined in priori. As an example of the uncertain yields in sawmills, consider the cutting pattern (*s*) that can produce 6 products (P1, P2, P3, P4, P5, P6) after sawing the logs of class (*c*). Table 1 represents four scenarios among all possible scenarios for the uncertain yields of this process.

Table 1. Scenarios for yields of a process in a sawing unit

Scenarios	Products					
	P1	P2	P3	P4	P5	P6
1	1	0	1	0	1	1
2	2	1	1	0	1	0
3	1	0	0	1	1	1
4	2	0	0	1	0	1

Uncertainty in the market demand for different lumbers is another important parameter that should be taken into account in sawmill production planning. We focus on operational level production planning in a sawing unit. The decision variables include the optimal quantity of log consumption from different classes and selection of best cutting patterns for each log class in each period of the planning horizon, in order to fulfill the demand. The objective is to minimize log consumption cost, as well as products inventory and backorder costs. Regarding to the potential significance of yield and demand uncertainty on the production plan, and customer orientation which is at center of attention in the sawmills which are dependent on the export markets, obtaining production plans with minimum expected backorder size is an important goal of production planning in sawmills.

Different approaches have been already proposed in the literature to address sawmill production planning. The first approach is focused on combined optimization type solutions linked to real-time simulation sub-systems (Mendoza et al., 1991; Maness and Adams, 1991; Maness and Norton, 2002). In this approach, the stochastic characteristics of logs are taken into account by assuming that all the input logs are scanned through an X-ray scanner, before planning. Maness and Norton (2002) developed an integrated multi-period production planning model which is the combination of an LP model and a log sawing optimizer (simulator). The LP model acts as a coordinating problem that allocates limited resources. The log sawing optimization models are used to generate columns for the coordinating LP based on the products' shadow prices. Although the stochastic characteristics of logs are considered in this approach, it includes the following limitations to be implemented: logs, needed for the next planning horizon, are not always available in the sawmill to be scanned before planning. Furthermore, to implement this method, the logs should be processed in production line in the same order they have been simulated, which is not an easy practice. Finally scanning logs before planning is a time consuming process in high capacity sawmills which delays the planning process. In the second approach, the randomness of the processes yields as well as demand is simplified and their expected value is considered in a MPMP linear programming model (Gaudreault et al., 2004). However, the production plans issued by these models result usually in extra inventory of products with lower quality and price while backorder of products with higher quality and price. In Kazemi et al. (2008a) a two-stage stochastic program with recourse is proposed to address sawmill production planning by considering the random yield. The solutions of stochastic model are considerably superior to those of deterministic model in terms of the expected inventory and backorder costs. Among different contributions in the literature for sawmill production planning, we did not succeed to find any contribution that considers simultaneously the random demand and yield. In the next section, the computational results of implementing the proposed multi-stage stochastic program for a realistic scale sawmill example is provided.

5. Computational results

In this section, we report on computational experiment with the proposed multi-stage stochastic programming approach for a realistic scale sawmill. The objective of our experiment is to investigate the quality of production plans suggested by multi-stage stochastic programming comparing to those of deterministic LP, and two-stage stochastic programming. We also compute the value of multi-stage stochastic programming (VMSP) for this example. In the following, we first describe our experimental environment and then report on the experimental results in the light of the mentioned objectives.

5.1. Experimental environment

A prototype sawmill is selected to illustrate the application of the multi-stage stochastic model. The prototype sawmill is a typical medium capacity softwood sawmill located in Quebec (Canada). The sawmill focuses on sawing high-grade products to the domestic markets as well as export products to the USA. It is assumed that the input bucked logs into the sawing unit are categorized into 3 classes. 5 different cutting patterns are available. The sawing unit produces 27 products of custom sizes (e.g. 2(in)×4(in), 2(in)×6(in) lumbers) in four lengths. In other words, there are 15 processes all can produce 27 products with random yields. We consider two bottleneck machines: Trimmer and Bull. The planning horizon consists of 30 periods (days). It would be worth mentioning that the data used in this example are based on the gathered data from different sawmills in Quebec province (Canada). As the list of custom sizes, machine parameters and prices are proprietary, they are not reported in this paper. The hybrid scenario tree for uncertain demand and yield in this example is generated as follows.

Demand Scenario tree

At each stage of scenario tree, except stage 1, based on the historical data for products demands (per day) in Quebec sawmill, we estimate a normal distribution for demand. We consider the same probability distribution for all the products. The normal distribution is then approximated by a 3 point discrete distribution by using Gaussian quadrature method (Miller and Rice, 1983). Since considering each time period as a stage leads to an extremely large number of scenarios, we need to approximate the scenario tree by something more manageable. In our computational experiment, we supposed that the demand for the next 10 days has a stationary behavior, which is a realistic assumption in the lumber market. Thus, we clustered the 30 periods planning horizon into 3 stages and hence the multi-stage decision process is approximated by a 4-stage one. The first stage consists of time period zero (present time), the second-stage includes periods 1-10, etc. The mentioned approximations results a scenario tree including 27 demand scenarios and 40 nodes. We consider three different normal distributions for demand with the same mean but different variances (5% mean, 20% mean, and 30% mean). Thus, three demand trees (DT1, DT2, DT3) and a total of 3 test problems are considered. In all the test problems we consider the same distribution for stages 2 to 4. However, different distributions can be considered for each stage without adding to the complexity of the resulted multi-stage model.

Yield Scenarios

As we mentioned in section 3, at each node of demand scenario tree a number of yield scenarios are taken into account. These scenarios are generated as follows. Based on the historical data in Quebec sawmills for the yields of processes, a normal distribution was estimated for the random yields (see Kazemi et al. 2007, 2008a, b). The normal distribution corresponding to the yield of each process was then approximated by three scenarios, by using Gaussian quadrature method (Miller and Rice, 1983). As the randomness of processes yields is the result of non-homogeneity in quality of logs, we consider three scenarios for yield of each log class. As we considered 3 classes of logs in this example, the total number of yield scenarios is equal to $3^3 = 27$. It should be noted that the same yield scenarios are considered in the three test problems.

The above scenario generation approach for uncertain demand and yield in this sawmill production planning example results a hybrid scenario tree similar to the one in figure 2 with 40 nodes, where each node includes 3 branches as demand scenarios and 27 branches as yield scenarios. The total number of scenarios at the end of stage 4 is equal to 27×27=729. The compact multi-stage stochastic model (7)-

(11) for this sawmill example is a linear programming (LP) model with nearly 600000 decision variables and 300000 constraints.

CPLEX 10 and OPL 5.1 are used to solve the linear program (7)-(11) and to perform further analysis on the solutions of the test problems. All numerical experiments are conducted on an AMD Athlon™ 64×2 dual core processor 3800+, 2.01 GHz, 3.00 GB of RAM, running Microsoft Windows Server 2003, standard edition.

5.2. Quality of multi-stage stochastic model solution

In this section, for the three test problems mentioned in 5.1, we compare the solution of 4-stage stochastic programming model to those of a 3-stage, and 2-stage stochastic programming model as well as mean-value deterministic model. It should be noted that in the 3-stage model, the 30 periods planning horizon is clustered into 2 stages, each includes 15 periods. In other words, in order to reduce the size of the multi-stage model, it was supposed that the random demand has a stationary behavior during each 15 days. The 2-stage stochastic model corresponds to considering a static probability distribution for the uncertain demand during the planning horizon. In table 2 the solutions of mentioned models for the three test problems are compared with respect to the expected total cost, the expected material consumption cost, as well as the expected inventory and backorder costs. It should be noted that the expected inventory/backorder costs of 3-stage, 2-stage, and mean-value deterministic models are computed by setting the production plan variables (X_{at}) in the 4-stage stochastic model (7)-(11) as the optimal production plan (X^*_{at}) proposed by the mentioned models. In other words, the expected inventory/backorder costs of production plans proposed by the 3-stage, 2-stage and deterministic model are computed for the hybrid 4-stage scenario tree corresponding to the uncertain yield and demand in each test problem. As it can be observed in table 2, in all the tree test problems the solution of 4-stage stochastic model is significantly superior to those of deterministic model. Furthermore, if the uncertain demand is considered as a random variable with a static probability distribution during the planning horizon (as in the two-stage stochastic programming model), the expected material cost as well as the expected inventory/backorder costs of the production plan are considerably higher than those of multi-stage stochastic model's plan. Finally, by clustering the planning horizon into two stages (as in 3-stage stochastic programming model) the expected inventory/backorder costs of the plan are higher than those of 4-stage stochastic model. The last column of table 2 indicates that the high quality of multi-stage stochastic model owns higher computational time compared to those of deterministic and two-stage ones.

Table 2- Cost comparison of different production planning models

Demand tree	Production planning model	Expected total cost	Expected material cost	Expected inventory/backorder costs	CPU time (minutes)
DT1	4-Stage SLP	1957950	1737500	220450	29
	3-Stage SLP	1973563	1746415	227148	6
	2-Stage SLP	2118125	1788142	329983	2
	Mean-value deterministic LP	2129030	1751675	377355	0
DT2	4-Stage SLP	2028777	1749677	279100	29
	3-Stage SLP	2038543	1756840	281703	6
	2-Stage SLP	2391261	1889265	501996	2
	Mean-value deterministic LP	2432717	1751675	681042	0
DT3	4-Stage SLP	2163458	1766387	397071	29
	3-Stage SLP	2206165	1763839	442326	6
	2-Stage SLP	2836707	2025122	811585	2
	Mean-value deterministic LP	3095556	1751675	1343881	0

Figures 5 and 6 illustrate better the comparison between the total expected cost as well as expected inventory/backorder costs of different models for three test problems which are distinguished by variability of demand at each stage. As the variability of demand increases at each stage, the difference between the expected cost of multi-stage stochastic model's plan and deterministic and two-stage stochastic models' plans increases. In other words, the significance of using a multi-stage programming model instead of a two-stage or deterministic model is increased as the variability of demand increases at each stage of scenario tree. It would be worth mentioning that by increasing the number of stages in the demand scenario tree, which is equivalent to reducing the number of periods at each stage, the uncertain behavior of demand can be captured more precisely. Thus, a production plan with lower expected cost can be obtained. However, as the difference between the expected cost of 4-stage and 3-stage models in not very significant in the three test problems (see figures 5 and 6), we did not consider more stages in the scenario trees in the test problems.

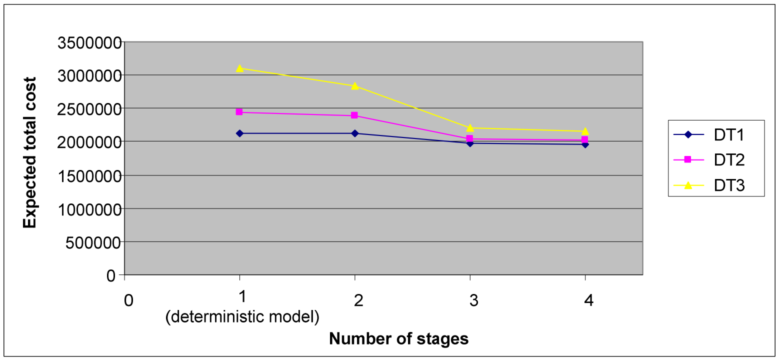

Figure 5. Expected total cost comparison of different production planning models

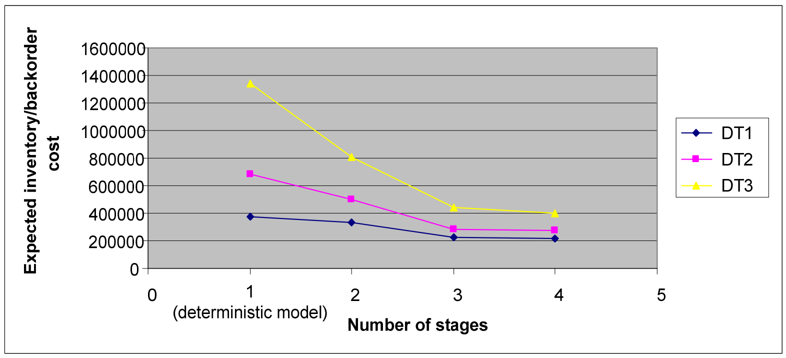

Figure 6. Expected inventory/backorder costs comparison of different production planning models

5.3. Value of multi-stage stochastic programming

As it was mentioned in section 3, we considered the production plan (X_{at}) as full recourse with respect to demand scenarios. In other words, we assumed a flexible production plan that can be adjusted based on the demand scenarios, at different stages. However, in some manufacturing environments the production plan is not flexible and should be fixed at the beginning of planning horizon. Thus, a simple recourse multi-stage stochastic model should be used to determine the plan. In this section, we compare the solutions of multi-stage stochastic programs with full recourse and simple recourse, for the three test problems. In table 3, it can easily be verified that in all the test problems the total cost of full recourse problem is smaller than that of the simple recourse problem. This should come as no surprise, since the multi-stage model with full recourse offers more flexibility in the production plan decisions with respect to the uncertain states of demand. We denote the optimal objective values corresponding to full recourse and simple recourse multi-stage stochastic programs by v^{FR} , and v^{SR} , respectively. The value of multi-stage stochastic programming (VMSP) is defined as follows (Huang and Shabbir, 2005; Huang 2005): $VMSP = v^{SR} - v^{FR}$.

Table 3- Value of multi-stage stochastic programming in the three test problems

Demand tree	4-stage stochastic model	Objective function value	VMSP
DT1	Full recourse	1957950	79195
	Simple recourse	1973563	
DT2	Full recourse	2028777	174076
	Simple recourse	2037145	
DT3	Full recourse	2163458	300000
	Simple recourse	2202853	

Value of multi-stage stochastic programming (VMSP) indicates the value of allowing the production plan to be adjusted for different scenarios at each stage of decision process instead of fixing its value at the beginning of planning horizon. Figure 7 compares the VMSP of the three test problems with different variability levels in demand. As it can be observed in figure 7, the value of multi-stage stochastic solution increases with the variability of demand. In other words, as the variability of demand increases at each stage, considering a full recourse multi-stage stochastic model becomes more significant.

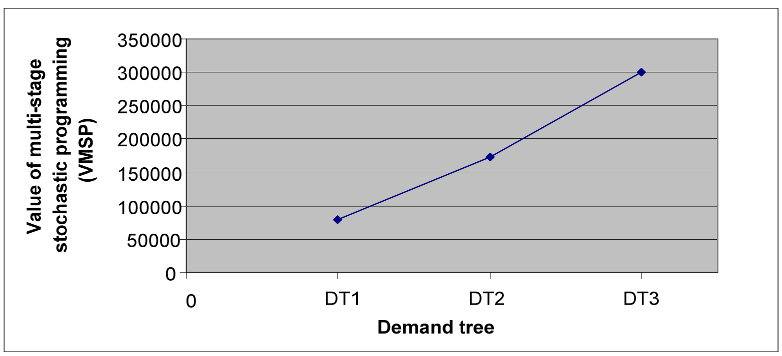

Figure 7. VMSP comparison of different test problems with different demand variability

6. Conclusions

In this paper, we addressed a multi-period, multi-product (MPMP) production planning problem under uncertainty in products demands and processes yields. We proposed a multi-stage stochastic model to address the problem. The uncertain demand was modeled as a dynamic stochastic process presented as a scenario tree. The uncertain yield was modeled as a static random variable with a stationary probability distribution during the planning horizon. We integrated the uncertain yield and demand into a hybrid scenario tree. The proposed approach was applied for sawmill production planning under the uncertainty in raw material (log) quality and product (lumber) demand. We presented the computational results using a realistic scale prototype sawmill. Our numerical results indicated that the quality of 4-stage stochastic model solutions is significantly higher than those of the mean-value deterministic and two-stage stochastic models. Moreover, it was shown that as the variability of demand is augmented at each stage of the scenario tree, the significance of using the multi-stage stochastic programming approach is increased. As further extensions of this work, we can consider seasonal demand and different trends at each stage of demand scenario tree. Moreover, the proposed approach can be applied for production planning in other manufacturing environments with uncertain demand and non-homogeneous and random characteristics of raw materials which results the random processes yields.

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